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AUTHOR(S):

Yoshida, Masamichi

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Denjoy odometer with cut number 1 or 2

By

Masamichi YOSHIDA*

Abstract

We construct a new class of numeration systems which properly includes the class of dual Ostrowski numeration systems and whose associated odometers are topologically conjugate to Denjoy systems with cut number 1 or 2.

§ 1. Introduction

The main aim of this paper is a generalization of dual Ostrowski numeration system and its associated odometer. All statements in this section are proved later in a more general setup.

Let $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\mathbb{B} = (0, 1) \setminus \mathbb{Q}$. The Gauss map $G : \mathbb{B} \to \mathbb{B}$ is defined by $G(\alpha) = \{\frac{1}{\alpha}\}$ (the fractional part of $\frac{1}{\alpha}$). It is well-known that G generates the simple continued fraction expansion of α: precisely, letting $\alpha_n = G^n(\alpha)$ and $a_n = \lfloor \frac{1}{\alpha_n} \rfloor$, we have

$$\alpha = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}}.$$

Set $M^\alpha = \{x = x_0 x_1 x_2 \cdots \in \prod_{n \in \mathbb{N}_0} \{0, 1, \cdots, a_n\} \mid x_n = a_n \Longrightarrow x_{n+1} = 0\}$. It is also well-known that for any $\xi_0 \in [0, 1]$ there is $x \in M^\alpha$ with

$$\xi_0 = \nu^\alpha(x) := \sum_{n \in \mathbb{N}_0} x_n \prod_{j=0}^{n} \alpha_j$$

by using usual *greedy algorithm*, that is, setting $x_n = \lfloor \frac{\xi_n}{\alpha_n} \rfloor$ and $\xi_{n+1} = \{\frac{\xi_n}{\alpha_n}\}$. This expansion of ξ_0 is called the **dual Ostrowski expansion** of ξ_0 based on α. See Subsection 6.4.3 of [3]. Moreover, we can see that for any $x \in M^\alpha$, the series $\nu^\alpha(x)$ converges and $\nu^\alpha(x) \in [0, 1]$. Denote by $\{\nu^\alpha\}(x)$ the fractional part of $\nu^\alpha(x)$ and so we have a surjective map

$$\{\nu^\alpha\} : M^\alpha \to [0, 1).$$

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*Deparment of Mathematics, Osaka City University, 3-3-138 Sugimoto Sumiyoshi-ku, Osaka-shi, 558-8585, Japan.
email:yoshida@sci.osaka-cu.ac.jp

On the other hand, we have an "odometer"

$$H_\alpha : M^\alpha \to M^\alpha$$

in a natural way and call H_α the *dual Ostrowski odometer* on M^α. The formal definition of H_α is as follows. Define $c = a_0 0 a_2 0 \cdots$. For each $c \neq x \in M^\alpha$, let

$$L(x) = \min\{n \in \mathbb{N}_0 \mid x_n \neq c_n\}.$$

Note $L(x)$ is even. Define $H_\alpha(c) = 0a_1 0a_3 0 \cdots$ and for each $c \neq x \in M^\alpha$ with $L = L(x)$

$$H_\alpha(x) = \begin{cases} 0a_1 0a_3 \cdots 0a_{L-3} 0(a_{L-1}-1)(x_L+1)x_{L+1}x_{L+2}\cdots \\ \qquad\qquad \text{if } x_L < a_L - 1 \text{ or if } x_L = a_L - 1 \text{ and } x_{L+1} = 0 \\ 0a_1 0a_3 \cdots 0a_{L-1} 0(x_{L+1}-1)x_{L+2}x_{L+3}\cdots \\ \qquad\qquad \text{otherwise.} \end{cases}$$

It is easy to check $H_\alpha(x) \in M^\alpha$. At first sight, the definition of H_α may look artificial, but it is natural under "*carry operation*". See the proof of Lemma 7.7 and its subsequent discussion. There is the following theorem:

(1) $\{\nu^\alpha\}$ is at most 2-to-1 and H_α is a homeomorphism with $\{\nu^\alpha\} \circ H_\alpha = R_\alpha \circ \{\nu^\alpha\}$ where $R_\alpha : [0,1) \to [0,1)$ is the rotation with angle α.

(2) $\{\xi \in [0,1) \mid \sharp\{\nu^\alpha\}^{-1}(\xi) = 2\} = \mathcal{O}_\alpha$ where $\mathcal{O}_\eta$ is the orbit of $\eta \in [0,1)$ under R_α, that is, $\mathcal{O}_\eta = \{R_\alpha^n(\eta) \mid n \in \mathbb{Z}\}$

(3) $\mathbf{e} \circ \nu^\alpha : M^\alpha \to S^1$ is continuous where $\mathbf{e}(\eta) = \exp(2\pi i \eta)$ and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Fact (2) says that the points, which have 2-way expansions in M^α, form a single orbit $\mathcal{O}_\alpha$. Under usual identification of R_α with $\mathbf{e} \circ R_\alpha \circ (\mathbf{e}|_{[0,1)})^{-1} : S^1 \to S^1$, (1) and (3) say that H_α is an at most 2-to-1 topological extension of $R_\alpha : S^1 \to S^1$. Moreover, this theorem implies that H_α is topologically conjugate to a *Denjoy system with rotation number α and cut number* 1. In other words, H_α is an odometer model for a Denjoy system with rotation number α and cut number 1. See Section 8 for definitions of Denjoy system, rotation number and cut number.

In this paper, when $\alpha \in \mathbb{B}$ and $\beta \in [0,1)$ are given, we address a generalization of this theorem: that is, to construct a numeration system $\nu^{\alpha,\beta}$ such that the points, which have 2-way expansions, form $\mathcal{O}_\alpha \cup \mathcal{O}_\beta$ and an odometer $H_{\alpha,\beta}$ associated with $\nu^{\alpha,\beta}$ is topologically conjugate to a Denjoy system with rotation number α and cut number 1 or 2 (Theorems 1.1 and 1.2). In [1], Cortez and Rivera-Letelier showed a general model theorem (up to topological orbit equivalence) for the class of uniquely ergodic Cantor minimal (dynamical) systems, by using inverse limits of generalized odometers. More directly than [1], we shall construct an odometer model for the small subclass of Denjoy systems with rotation number α and cut number 1 or 2, without using inverse limit. Especially the odometer in this paper is a bijection.

Instead of the Gauss map G, we shall begin with $T : \mathbb{B} \times [0,1) \to \mathbb{B} \times [0,1)$ defined by

$$T(\alpha,\beta) = \begin{cases} \left(\left\{\dfrac{1}{\alpha}\right\}, \left\{\dfrac{-\beta}{\alpha}\right\}\right) \text{ if } \left\{\dfrac{-1}{\alpha}\right\} \geq \left\{\dfrac{-\beta}{\alpha}\right\} \\ \left(\left\{\dfrac{-1}{\alpha}\right\}, \left\{\dfrac{\beta}{\alpha}\right\}\right) \text{ otherwise.} \end{cases}$$

(cf. This map T is a modification of a map used in [2], Théorème 3.2, pp. 299-300.) Note $T(\alpha,0)=(G(\alpha),0)$ so T is an extension of G. Define $\iota:\mathbb{B}\times[0,1)\to\{0,1\}$ by

$$\iota(\alpha,\beta)=\begin{cases}0\text{ if }\{\frac{-1}{\alpha}\}\geq\{\frac{-\beta}{\alpha}\}\\ 1\text{ otherwise.}\end{cases}$$

Letting $(\alpha_n,\beta_n)=T^n(\alpha,\beta),\ \iota_n=\iota(\alpha_n,\beta_n),\ a_n=\lfloor\frac{1}{\alpha_n}\rfloor+\iota_n$ and $b_n=\lceil\frac{\beta_n}{\alpha_n}\rceil$, set

$$M^{\alpha,\beta}=\left\{x=x_0x_1x_2\cdots\in\prod_{n\in\mathbb{N}_0}\{0,1,\cdots,a_n\}\ \middle|\ \begin{array}{l}x_n=0\implies x_{n+1}\geq_{\iota_n}b_{n+1}-\iota_n\\ x_n=a_n\implies x_{n+1}\leq_{\iota_n}b_{n+1}+\iota_n\end{array}\right\}$$

where the inequality $\geq_0$ (resp. $\geq_1$) means $\geq$ (resp. $\leq$).

In particular when $\beta=0$, we see that $\alpha_n=G^n(\alpha),\ \beta_n=0,\ \iota_n=0,\ a_n=\lfloor 1/G^n(\alpha)\rfloor$ and $b_n=0$ for each $n\in\mathbb{N}_0$, and hence $M^{\alpha,0}=M^\alpha$.

We propose a new numeration system $\nu^{\alpha,\beta}$ as follows. Define $\nu^{\alpha,\beta}:M^{\alpha,\beta}\to[0,1]$ by

$$\nu^{\alpha,\beta}(x)=\sum_{n\in\mathbb{N}_0}(-1)^{e_n}(x_n-(-1)^{\iota_n}\beta_{n+1})\prod_{j=0}^{n}\alpha_j$$

where $e_0=0$ and $e_{n+1}=|e_n-\iota_n|$. See Sections 3 and 5 for precise argument about $\nu^{\alpha,\beta}$. Note that $(-1)^{e_n}=(-1)^{\iota_0+\iota_1+\cdots+\iota_{n-1}}$ for each $n\geq 1$, because $(-1)^{e_{n+1}}=(-1)^{e_n}(-1)^{\iota_n}$.

In particular when $\beta=0$, we have $\nu^{\alpha,0}=\nu^\alpha$ (because $\iota_n=\beta_n=0$ and $\alpha_n=G^n(\alpha)$), that is, $\nu^{\alpha,\beta}$ is a generalization of dual Ostrowski numeration system.

On the other hand, we will show $\beta\notin\mathcal{O}_\alpha$ if and only if $0<b_n<a_n$ for each $n\geq 1$. See Proposition 7.13 in Section 7. Here, we give an example:

Example. Let $\alpha=\sqrt{2}-1$ and $\beta=\frac{1-\alpha}{2}=1-\frac{1}{\sqrt{2}}$. Since $\frac{1}{\alpha}=\sqrt{2}+1$ and $\frac{\beta}{\alpha}=1-\beta$, we have $\lfloor\frac{1}{\alpha}\rfloor=2,\ \{\frac{1}{\alpha}\}=\alpha,\ \lceil\frac{\beta}{\alpha}\rceil=1$ and $\{\frac{-\beta}{\alpha}\}=\beta<1-\alpha=\{\frac{-1}{\alpha}\}$. So $\iota(\alpha,\beta)=0$ and $T(\alpha,\beta)=(\alpha,\beta)$. Hence $\alpha_n=\alpha,\ \beta_n=\beta,\ \iota_n=0,\ a_n=2$ and $b_n=1$ for each $n\in\mathbb{N}_0$. So we have

$$M^{\alpha,\beta}=\left\{\ x\in\{0,1,2\}^{\mathbb{N}_0}\ \middle|\ \begin{array}{l}x_n=0\implies x_{n+1}\geq 1\\ x_n=2\implies x_{n+1}\leq 1\end{array}\right\},$$

in other words, $M^{\alpha,\beta}=\{x\in\{0,1,2\}^{\mathbb{N}_0}\mid x_nx_{n+1}\neq 00,22\ \text{ for each } n\in\mathbb{N}_0\}$. Moreover

$$\nu^{\alpha,\beta}(x)=\sum_{n=0}^{\infty}(x_n-\beta)\alpha^{n+1}=-\frac{\alpha}{2}+\sum_{n=0}^{\infty}x_n\alpha^{n+1}.$$

Concluding this section, we will have main theorems. For each $(\alpha,\beta)\in\mathbb{B}\times[0,1)$, we have an odometer,

$$H_{\alpha,\beta}:M^{\alpha,\beta}\to M^{\alpha,\beta},$$

which is natural under carry operation. See Section 7 for the definition of $H_{\alpha,\beta}$. Denote by $\{\nu^{\alpha,\beta}\}(x)$ the fractional part of $\nu^{\alpha,\beta}(x)$.

Theorem 1.1. *Let* $(\alpha,\beta)\in\mathbb{B}\times[0,1)$. *Then we have the following*:

(1) $\{\nu^{\alpha,\beta}\}: M^{\alpha,\beta}\to[0,1)$ *is an at most 2-to-1 surjection and* $H_{\alpha,\beta}: M^{\alpha,\beta}\to M^{\alpha,\beta}$ *is a homeomorphism with* $\{\nu^{\alpha,\beta}\}\circ H_{\alpha,\beta}=R_\alpha\circ\{\nu^{\alpha,\beta}\}$

(2) $\{\xi\in[0,1)\mid \sharp\{\nu^{\alpha,\beta}\}^{-1}(\xi)=2\}=\mathcal{O}_\alpha\cup\mathcal{O}_\beta$

(3) $\mathbf{e}\circ\nu^{\alpha,\beta}: M^{\alpha,\beta}\to S^1$ *is continuous.*

Theorem 1.2. *If* $(\alpha,\beta)\in\mathbb{B}\times[0,1)$, $\varphi_X: X\to X$ *is a Denjoy system with rotation number* α, *and the set of double points of a factor map* $F_X: X\to S^1$ *coincides* $\mathcal{O}_\alpha\cup\mathcal{O}_\beta$ *under the identifcation* $[0,1)$ *with* S^1 *via* $\mathbf{e}|_{[0,1)}$, *then there is a homeomorphism* $\psi: X\to M^{\alpha,\beta}$ *such that* $\psi\circ\varphi_X=H_{\alpha,\beta}\circ\psi$ *and* $F_X=\mathbf{e}\circ\nu^{\alpha,\beta}\circ\psi$.

See Section 8 for definitions of a *factor map* $F_X: X\to S^1$ and a *double point* of F_X where $\varphi_X: X\to X$ is a Denjoy system.

§ 2. Algorithm T

We study the property of $T:\mathbb{B}\times[0,1)\to\mathbb{B}\times[0,1)$ and the sequences $(\alpha_n,\beta_n)=T^n(\alpha,\beta)$, $\iota_n=\iota(\alpha_n,\beta_n)$, $a_n=\left\lfloor\frac{1}{\alpha_n}\right\rfloor+\iota_n$ and $b_n=\left\lceil\frac{\beta_n}{\alpha_n}\right\rceil$ when $(\alpha,\beta)\in\mathbb{B}\times[0,1)$ is given.

We begin with simple remarks. Note $\{-\xi\}=1-\{\xi\}$ for any $\xi\in\mathbb{R}$ with $\{\xi\}>0$. So we have

Remark 2.1. $\iota(\alpha,\beta)=1 \iff 0<\left\{\frac{\beta}{\alpha}\right\}<\left\{\frac{1}{\alpha}\right\}$.

Remark 2.2.

$$\left\{\frac{1}{\alpha_n}\right\}=\iota_n+(-1)^{\iota_n}\alpha_{n+1}$$
$$\left\{\frac{-\beta_n}{\alpha_n}\right\}=\iota_n+(-1)^{\iota_n}\beta_{n+1}.$$

Since $\xi=\lfloor\xi\rfloor+\{\xi\}=\lceil\xi\rceil-\{-\xi\}$ for any $\xi\in\mathbb{R}$, we obtain the fundamental equations:

Recursive equations (1) $\dfrac{1}{\alpha_n}=a_n+(-1)^{\iota_n}\alpha_{n+1}$

(2) $\dfrac{\beta_n}{\alpha_n}=b_n-\iota_n-(-1)^{\iota_n}\beta_{n+1}.$

By Remark 2.1 and the definition of T, we have

Remark 2.3. *If* $\iota(x,y)=1$ *then* $T(x,y)\in\{(z,w)\mid z\in\mathbb{B},\ 0<w<1-z\}$. *In general,* $T(\mathbb{B}\times[0,1))\subset\{(z,w)\mid z\in\mathbb{B},\ 0\le w\le 1-z\}$.

Lemma 2.4.

$$\left\lceil\frac{\beta}{\alpha}\right\rceil+\left\lceil\frac{1-\beta}{\alpha}\right\rceil=\left\lceil\frac{1}{\alpha}\right\rceil+\iota(\alpha,\beta)$$

$$\left\{\frac{-\beta}{\alpha}\right\}+\left\{\frac{\beta-1}{\alpha}\right\}=\left\{\frac{-1}{\alpha}\right\}+\iota(\alpha,\beta).$$

Proof. Note
$$\tfrac{1-\beta}{\alpha} = \tfrac{1}{\alpha} - \tfrac{\beta}{\alpha} = \lceil \tfrac{1}{\alpha} \rceil - \lceil \tfrac{\beta}{\alpha} \rceil - \Big(\{ \tfrac{-1}{\alpha} \} - \{ \tfrac{-\beta}{\alpha} \} \Big).$$
When $\{ \frac{-1}{\alpha} \} - \{ \frac{-\beta}{\alpha} \} \geq 0$ (i.e. $\iota(\alpha, \beta) = 0$), we have the desired one. Suppose $\iota(\alpha, \beta) = 1$. Then
$$-1 < \left\{ \frac{-1}{\alpha} \right\} - \left\{ \frac{-\beta}{\alpha} \right\} < 0$$
and so
$$\left\{ \frac{-(1-\beta)}{\alpha} \right\} = 1 + \left\{ \frac{-1}{\alpha} \right\} - \left\{ \frac{-\beta}{\alpha} \right\} \text{ and } \left\lceil \frac{1-\beta}{\alpha} \right\rceil = \left\lceil \frac{1}{\alpha} \right\rceil - \left\lceil \frac{\beta}{\alpha} \right\rceil + 1.$$
□

Moreover we state two lemmas:

Lemma 2.5. *For each $n \in \mathbb{N}_0$, there are $q, p \in \mathbb{Z}$ such that* $\displaystyle\prod_{j=0}^{n} \alpha_j = q\alpha + p$.

Lemma 2.6. $\displaystyle\lim_{n\to\infty} \prod_{j=0}^{n} \alpha_j = 0.$

In Appendix, we give the proof of these lemmas. We will use Lemma 2.6 in such a way that if $\{r_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}$ is bounded, then $\displaystyle\lim_{n\to\infty} r_n \prod_{j=0}^{n} \alpha_j = 0$.

For convenience' sake, put
$$\iota_{-1} = 0.$$
We list the property of (a_n, b_n, ι_n):

Proposition 2.7.

(1) *For each $n \in \mathbb{N}_0$, $\iota_{n-1} \leq b_n \leq a_n - \iota_{n-1}$*

(in other words, $\{b_n - \iota_{n-1},\ b_n + \iota_{n-1}\} \subset \{0, 1, \cdots, a_n\}$).

(2) *If there is $K \in \mathbb{N}_0$ such that $b_K = 0$, then $b_{K+1} = 0$.*

(3) *If there is $K \geq 1$ such that $b_K = a_K$, then $b_{K+1} = a_{K+1}$.*

(4) *If there is $K \in \mathbb{N}_0$ such that $\iota_n = 1$ ($\forall n \geq K$), then there are $k, l \geq K+1$ such that $b_k \neq 1$ and $b_l \neq a_l - 1$.*

Proof. By Lemma 2.4, for each $n \in \mathbb{N}_0$
$$b_n = \left\lceil \frac{\beta_n}{\alpha_n} \right\rceil = a_n + 1 - \left\lceil \frac{1-\beta_n}{\alpha_n} \right\rceil.$$
(1) Since $\left\lceil \frac{\beta_n}{\alpha_n} \right\rceil \geq 0$ and $\left\lceil \frac{1-\beta_n}{\alpha_n} \right\rceil \geq 1$ (because $0 \leq \beta_n < 1$), we have $0 \leq b_n \leq a_n$. Furthermore if $\iota_{n-1} = 1$, then $0 < \beta_n < 1 - \alpha_n$ by Remark 2.3, hence $1 \leq b_n \leq a_n - 1$.
(2) Note $T(\alpha, 0) = (G(\alpha), 0)$ for any $\alpha \in \mathbb{B}$. Suppose $b_K = 0$. Then $\beta_K = 0$. So since

$(\alpha_{K+1}, \beta_{K+1}) = T(\alpha_K, \beta_K) = (\alpha_{K+1}, 0)$, we have $b_{K+1} = 0$.
(3) Note $T(\alpha,\ 1-\alpha) = (G(\alpha),\ 1 - G(\alpha))$ for any $\alpha \in \mathbb{B}$. Suppose $b_K = a_K$ for some $K \geq 1$. Then $\left\lceil \frac{1-\beta_K}{\alpha_K} \right\rceil = 1$. Moreover $1 - \beta_K = \alpha_K$, because $\beta_K \leq 1 - \alpha_K$ by Remark 2.3. So since $(\alpha_{K+1}, \beta_{K+1}) = T(\alpha_K, \beta_K) = (\alpha_{K+1}, 1 - \alpha_{K+1})$, we have $b_{K+1} = a_{K+1}$.
(4) By recursive equation (2)

$$\beta_n = (b_n - \iota_n)\alpha_n - (-1)^{\iota_n}\beta_{n+1}\alpha_n.$$

Notice that

$$1 - \alpha_n - \beta_n = (a_n - b_n - \iota_n)\alpha_n - (-1)^{\iota_n}(1 - \alpha_{n+1} - \beta_{n+1})\alpha_n$$

(indeed, $1 - \alpha_n - \beta_n = a_n\alpha_n + (-1)^{\iota_n}\alpha_{n+1}\alpha_n - (2\iota_n + (-1)^{\iota_n})\alpha_n - (b_n - \iota_n)\alpha_n + (-1)^{\iota_n}\beta_{n+1}\alpha_n$ by recursive equations (1), (2) and $(-1)^{\iota_n} = 1 - 2\iota_n$).

Now we prove (4) by contradiction. Suppose that $\iota_n = 1$ for any $n \geq K$. Then $0 < \beta_{K+1} < 1 - \alpha_{K+1}$ by Remark 2.3.

Assume that $b_n = 1$ for any $n \geq K + 1$. Then, by the above equations

$$\beta_{K+1} = \beta_{n+1} \prod_{j=K+1}^{n} \alpha_j \quad (\forall n \geq K + 1)$$

Taking $n \to \infty$, we have $\beta_{K+1} = 0$ by Lemma 2.6, contradicting $\beta_{K+1} > 0$.

Similarly, assume that $b_n = a_n - 1$ for any $n \geq K + 1$. Then, by the above equations

$$1 - \alpha_{K+1} - \beta_{K+1} = (1 - \alpha_{n+1} - \beta_{n+1}) \prod_{j=K+1}^{n} \alpha_j \quad (\forall n \geq K + 1)$$

Taking $n \to \infty$, we have $1 - \alpha_{K+1} - \beta_{K+1} = 0$ by Lemma 2.6, contradicting $\beta_{K+1} < 1 - \alpha_{K+1}$. □

By Proposition 2.7 (1), (2) and (3), we have

Remark 2.8.

If there is $K \in \mathbb{N}_0$ such that $b_K = 0$, then $b_n = 0$ $(\forall n \geq K)$ and $\iota_n = 0$ $(\forall n \geq K - 1)$.

If there is $K \geq 1$ such that $b_K = a_K$, then $b_n = a_n$ $(\forall n \geq K)$ and $\iota_n = 0$ $(\forall n \geq K - 1)$.

In particular, for each $K \geq 1$, we have $b_K \in \{0, a_K\} \implies \iota_K = 0$.

§ 3. (α, β)-Markovian numeration system

For each $i \in \{0, 1\}$ and $\xi, \eta \in \mathbb{R}$, define

$$\xi \leq_i \eta \iff (-1)^i\xi \leq (-1)^i\eta.$$

Thus $\leq_0$ is the usual inequality $\leq$, and $\leq_1$ is the inequality $\geq$.

From now on, let $(\alpha, \beta) \in \mathbb{B} \times [0, 1)$ be arbitrarily fixed. First we define (α, β)-Markovian sequences:

Definition 3.1 (Markovian space)**.** Let $x = x_0x_1x_2\cdots \in \prod_{n\in\mathbb{N}_0}\{0,1,\cdots,a_n\}$. We say that x is (α,β)-Markovian if x satisfies the following conditions, $(1)_n, (2)_n$, for each $n \in \mathbb{N}_0$:

$$\begin{aligned}&(1)_n\; x_n = 0 \implies x_{n+1} \geq_{\iota_n} b_{n+1} - \iota_n\\ &(2)_n\; x_n = a_n \implies x_{n+1} \leq_{\iota_n} b_{n+1} + \iota_n.\end{aligned}$$

Denote by M (or $M^{\alpha,\beta}$) the set of (α,β)-Markovian sequences.

We always use the 0-1 sequnce $e_0e_1e_2\cdots$ defined by

$$e_0 = 0,\; e_{n+1} = |e_n - \iota_n|$$

and the following simple formula

$$(-1)^{e_{n+1}} = (-1)^{e_n}(-1)^{\iota_n}.$$

Write

$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

Simply note $e_n = 0 \iff e_{n+1} = \iota_n$ (or equivalently, $e_n = 1 \iff \overline{e_{n+1}} = \iota_n$). So we have

Remark 3.2. *Consider the following conditions*:

$$\begin{aligned}&(1')_n\; x_n = e_na_n \implies x_{n+1} \geq_{e_{n+1}} b_{n+1} - (-1)^{e_n}\iota_n\\ &(2')_n\; x_n = \overline{e_n}a_n \implies x_{n+1} \leq_{e_{n+1}} b_{n+1} - (-1)^{\overline{e_n}}\iota_n.\end{aligned}$$

In case $e_n = 0$, we see that $(1')_n$ is the same condition as $(1)_n$ in Definition 3.1, and $(2')_n$ is $(2)_n$; in case $e_n = 1$, we see that $(1')_n$ is $(2)_n$, and $(2')_n$ is $(1)_n$.

Definition 3.3. For each $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ define

$$\nu_n(k) = (-1)^{e_n}(k - (-1)^{\iota_n}\beta_{n+1})\prod_{j=0}^{n}\alpha_j,$$

and for each sequence $x = x_0x_1x_2\cdots$ define (formally)

$$\nu(x) = \nu^{\alpha,\beta}(x) = \sum_{n\in\mathbb{N}_0}\nu_n(x_n).$$

In Section 5, we will prove that for any $x \in M$ the series $\nu(x)$ converges in $[0,1]$. We call the map $\nu : M \to [0,1]$ the (α,β)-**numeration system**.

We prove if a sequence $z = z_0z_1z_2\cdots$ is *extremal* in the following sense, then $\nu(z)$ converges.

Definition 3.4 (Extremal sequences)**.** Let $z = z_0z_1z_2\cdots$ and $k \in \mathbb{N}_0$.
We call z a k-left extremal sequence if for each $n \geq k$,

$$z_n = \begin{cases} e_na_n & \text{if } n \equiv k \bmod 2\\ b_n - (-1)^{e_{n-1}}\iota_{n-1} & \text{otherwise.}\end{cases}$$

We call z a k-right extremal sequence if for each $n \geq k$,

$$z_n = \begin{cases} \overline{e_n} a_n & \text{if } n \equiv k \bmod 2 \\ b_n - (-1)^{\overline{e_{n-1}}} \iota_{n-1} & \text{otherwise.} \end{cases}$$

When z is k-left extremal (resp. k-right extremal) for some $k \in \mathbb{N}_0$, we say simply that z is left extremal (resp. right extremal). When z is left extremal or right extremal, we say simply that z is extremal.

(For example, when $\beta = 0$ (or equivalently, $b_0 = 0$), we have $\iota_n = b_n = e_n = 0$ $(\forall n)$ and so the 0-left extremal sequence is $0000\cdots$ and the 0-right extremal sequence is $a_0 0 a_2 0 \cdots$.)

We use the convention that the symbol $\prod_{j=0}^{-1} \alpha_j$ means 1.

Lemma 3.5. *If z is extremal then $\nu(z)$ converges. Moreover, the following statements hold*:

(1) *If z is k-left extremal, then* $\sum_{n=k}^{\infty} \nu_n(z_n) = -e_k \prod_{j=0}^{k-1} \alpha_j$.

(2) *If z is k-right extremal, then* $\sum_{n=k}^{\infty} \nu_n(z_n) = \overline{e_k} \prod_{j=0}^{k-1} \alpha_j$.

So since $e_0 = 0$, especially we have that if z is 0-left extremal then $\nu(z) = 0$; if z is 0-right extremal then $\nu(z) = 1$.

Note. We will prove the converse (in M) of (1), (2) in this lemma: see Proposition 5.2 in Section 5.

Proof. We show the following formula: for each $n \in \mathbb{N}_0$

$$(I)\ \nu_n(e_n a_n) + \nu_{n+1}(b_{n+1} - (-1)^{e_n} \iota_n) = -e_n \prod_{j=0}^{n-1} \alpha_j + e_{n+2} \prod_{j=0}^{n+1} \alpha_j$$

$$(II)\ \nu_n(\overline{e_n} a_n) + \nu_{n+1}(b_{n+1} - (-1)^{\overline{e_n}} \iota_n) = \overline{e_n} \prod_{j=0}^{n-1} \alpha_j - \overline{e_{n+2}} \prod_{j=0}^{n+1} \alpha_j.$$

We use recursive equations (1), (2) and $(-1)^{e_{n+1}} = (-1)^{e_n}(-1)^{\iota_n}$ and the following three simple formulas: for each $s, t \in \{0, 1\}$

$$(-1)^s s = -s$$
$$(-1)^s \overline{s} = \overline{s}$$
$$|s - t| = s + (-1)^s t.$$

Proof of the formula (I):

$$\begin{aligned}
&\frac{\nu_n(e_n a_n) + \nu_{n+1}(b_{n+1} - (-1)^{e_n}\iota_n)}{\prod_{j=0}^{n} \alpha_j} \\
&= -e_n a_n - (-1)^{e_{n+1}}\beta_{n+1} + (-1)^{e_{n+1}} b_{n+1}\alpha_{n+1} + \iota_n \alpha_{n+1} - (-1)^{e_{n+1}}(-1)^{\iota_{n+1}}\beta_{n+2}\alpha_{n+1} \\
&= -e_n a_n + \iota_n\alpha_{n+1} - (-1)^{e_{n+1}}\Big(\beta_{n+1} - b_{n+1}\alpha_{n+1} + (-1)^{\iota_{n+1}}\beta_{n+2}\alpha_{n+1}\Big) \\
&= -e_n\left(\frac{1}{\alpha_n} - (-1)^{\iota_n}\alpha_{n+1}\right) + \iota_n\alpha_{n+1} + (-1)^{e_{n+1}}\iota_{n+1}\alpha_{n+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by recursive equations (1) and (2))} \\
&= -\frac{e_n}{\alpha_n} + \Big((-1)^{\iota_n} e_n + \iota_n\Big)\alpha_{n+1} + (-1)^{e_{n+1}}\iota_{n+1}\alpha_{n+1} \\
&= -\frac{e_n}{\alpha_n} + |e_n - \iota_n|\alpha_{n+1} + \Big(|e_{n+1} - \iota_{n+1}| - e_{n+1}\Big)\alpha_{n+1} \\
&= -\frac{e_n}{\alpha_n} + (e_{n+1} + e_{n+2} - e_{n+1})\alpha_{n+1} \;=\; -\frac{e_n}{\alpha_n} + e_{n+2}\alpha_{n+1}.
\end{aligned}$$

In the same way, we show the formula (II):

$$\begin{aligned}
&\frac{\nu_n(\overline{e_n} a_n) + \nu_{n+1}(b_{n+1} - (-1)^{\overline{e_n}}\iota_n)}{\prod_{j=0}^{n} \alpha_j} \\
&= \overline{e_n} a_n - (-1)^{e_{n+1}}\beta_{n+1} + (-1)^{e_{n+1}} b_{n+1}\alpha_{n+1} - \iota_n \alpha_{n+1} - (-1)^{e_{n+1}}(-1)^{\iota_{n+1}}\beta_{n+2}\alpha_{n+1} \\
&= \overline{e_n} a_n - \iota_n\alpha_{n+1} - (-1)^{e_{n+1}}\Big(\beta_{n+1} - b_{n+1}\alpha_{n+1} + (-1)^{\iota_{n+1}}\beta_{n+2}\alpha_{n+1}\Big) \\
&= \overline{e_n}\left(\frac{1}{\alpha_n} - (-1)^{\iota_n}\alpha_{n+1}\right) - \iota_n\alpha_{n+1} + (-1)^{e_{n+1}}\iota_{n+1}\alpha_{n+1} \\
&= \frac{\overline{e_n}}{\alpha_n} - \Big((-1)^{\iota_n} \overline{e_n} + \iota_n\Big)\alpha_{n+1} + (-1)^{e_{n+1}}\iota_{n+1}\alpha_{n+1} \\
&= \frac{\overline{e_n}}{\alpha_n} - |\overline{e_n} - \iota_n|\alpha_{n+1} + \Big(|e_{n+1} - \iota_{n+1}| - e_{n+1}\Big)\alpha_{n+1} \\
&= \frac{\overline{e_n}}{\alpha_n} - (\overline{e_{n+1}} - e_{n+2} + e_{n+1})\alpha_{n+1} \;=\; \frac{\overline{e_n}}{\alpha_n} - \overline{e_{n+2}}\alpha_{n+1}.
\end{aligned}$$

Now we return to the proof of Lemma 3.5.

(1) Let z be k-left extremal. Then by formula (I), for each $N \geq k$ with $N \equiv k \bmod 2$

$$\sum_{n=k}^{N+1} \nu_n(z_n) = -e_k \prod_{j=0}^{k-1} \alpha_j + e_{N+2} \prod_{j=0}^{N+1} \alpha_j$$

and since

$$\nu_{N+2}(z_{N+2}) = \nu_{N+2}(e_{N+2} a_{N+2}) = -(e_{N+2} a_{N+2} + (-1)^{e_{N+3}}\beta_{N+3}) \prod_{j=0}^{N+2} \alpha_j,$$

we have

$$\begin{aligned}\sum_{n=k}^{N+2}\nu_n(z_n) &= -e_k\prod_{j=0}^{k-1}\alpha_j+\left(e_{N+2}\left(\frac{1}{\alpha_{N+2}}-a_{N+2}\right)-(-1)^{e_{N+3}}\beta_{N+3}\right)\prod_{j=0}^{N+2}\alpha_j\\ &= -e_k\prod_{j=0}^{k-1}\alpha_j+(e_{N+2}(-1)^{\iota_{N+2}}\alpha_{N+3}-(-1)^{e_{N+3}}\beta_{N+3})\prod_{j=0}^{N+2}\alpha_j\\ &\qquad\text{(by recursive equation (1)).}\end{aligned}$$

As $N\to\infty$, $\sum_{n=k}^{\infty}\nu_n(z_n)=-e_k\prod_{j=0}^{k-1}\alpha_j$ by Lemma 2.6. Similarly (2) can be proved. $\square$

Lemma 3.6. *Let $k\in\mathbb{N}_0$.*
If x is k-left or k-right extremal, then for each $n\geq k$, $x_n\in\{0,1,\cdots,a_n\}$ and x satisfies conditions $(1)_n$ and $(2)_n$ in Definition 3.1.

So, especially if x is 0-left or 0-right extremal, then $x\in M$.

Proof. Let x be k-left extremal and $n\geq k$.

If $n-k$ is even, then $x_n=e_na_n\in\{0,a_n\}$. If $n-k$ is odd, then $x_n=b_n-(-1)^{e_{n-1}}\iota_{n-1}\in\{b_n-\iota_{n-1},\ b_n+\iota_{n-1}\}\subset\{0,1,\cdots,a_n\}$ by Proposition 2.7.

When $n-k$ is even, the condition $(1')_n$ in Remark 3.2 holds. Consider the case $n-k$ is odd.

First we show x satisfies the condition $(2)_n$, that is, $x_n=a_n\implies x_{n+1}\leq_{\iota_n} b_{n+1}+\iota_n$. Suppose $x_n=a_n$. If $b_n=a_n$, then $b_{n+1}=a_{n+1}$ and $\iota_n=0$ by Proposition 2.7 (note $n\geq k+1\geq 1$), and so $x_{n+1}\leq b_{n+1}+\iota_n$. Suppose $b_n\leq a_n-1$. Then $e_{n-1}=1$, $\iota_{n-1}=1$ and $b_n=a_n-1$ because $b_n-(-1)^{e_{n-1}}\iota_{n-1}=x_n=a_n$. Hence $e_n=|e_{n-1}-\iota_{n-1}|=0$ and $e_{n+1}=|0-\iota_n|=\iota_n$. Now, since $x_{n+1}=\iota_na_{n+1}$, we see that if $\iota_n=0$ then $x_{n+1}=0\leq b_{n+1}+\iota_n$; if $\iota_n=1$ then $x_{n+1}=a_{n+1}\geq b_{n+1}+\iota_n$. Anyway $(2)_n$ holds.

Similarly we can show x satisfies the condition $(1)_n$. The proof in the case that x is k-right extremal is also similar. $\square$

Now, by Lemmas 3.5 and 3.6, we obtain typical examples of (α,β)-Markovian sequences:
(1) If x is 0-left extremal then $x\in M$ and $\nu(x)=0$.
(2) If x is 0-right extremal then $x\in M$ and $\nu(x)=1$.

Here note that $e_1a_1\leq_{\iota_0} b_1+\iota_0$ and $\overline{e_1}a_1\geq_{\iota_0} b_1-\iota_0$, by Proposition 2.7 and $e_1=\iota_0$. Suppose $\beta>0$ (or equivalently, $b_0\geq 1$).
(3) If x is 1-left extremal with $x_0=b_0$, then x satisifies condition $(2)_0$ (since $e_1a_1\leq_{\iota_0} b_1+\iota_0$) so $x\in M$ and moreover by recursive equation (2)

$$\nu(x)=\nu_0(b_0)+\sum_{n=1}^{\infty}\nu_n(x_n)=(b_0-(-1)^{\iota_0}\beta_1)\alpha_0-e_1\alpha_0=\beta.$$

(4) If x is 1-right extremal with $x_0=b_0-1$, then x satisifies condition $(1)_0$ (since $\overline{e_1}a_1\geq_{\iota_0} b_1-\iota_0$) so $x\in M$ and moreover by recursive equation (2)

$$\nu(x)=\nu_0(b_0-1)+\sum_{n=1}^{\infty}\nu_n(x_n)=(b_0-1-(-1)^{\iota_0}\beta_1)\alpha_0+\overline{e_1}\alpha_0=\beta.$$

See Lemma 7.2 in Section 7 for another example of (α, β)-Markovian sequences.

§ 4. (α, β)-expansion of a real number in $[0,1]$

In this section, we show

Proposition 4.1. *For each $\xi \in [0,1]$, there is $x \in M$ such that $\xi = \nu(x)$.*

For the proof, we use the following notation: Let $\xi \in \mathbb{R}$ and $i \in \{0,1\}$. Define

$$[\xi]_i = \begin{cases} \lfloor \xi \rfloor & \text{if } i = 0 \\ \lceil \xi \rceil - 1 & \text{if } i = 1 \end{cases} \quad \text{and} \quad \{\xi\}_i = \begin{cases} \{\xi\} & \text{if } i = 0 \\ 1 - \{-\xi\} & \text{if } i = 1. \end{cases}$$

Then we have $\xi = [\xi]_i + \{\xi\}_i$ and note that

$$\xi \in \Big[[\xi]_0, [\xi]_0 + 1\Big), \ 0 \le \{\xi\}_0 < 1$$

and

$$\xi \in \Big([\xi]_1, [\xi]_1 + 1\Big], \ 0 < \{\xi\}_1 \le 1.$$

Write $\Delta_n = \left\{ \dfrac{-\beta_n}{\alpha_n} \right\}$.

Proof of Proposition 4.1. Recall if z is the 0-right extremal sequence, then $z \in M$ and $\nu(z) = 1$ by Lemmas 3.5 and 3.6. Suppose $0 \le \xi < 1$. Let $\xi_0 = \xi$. Define x_n and ξ_{n+1} inductively by

$$x_n = \left[\frac{\xi_n}{\alpha_n} + \Delta_n \right]_{e_n} \quad \text{and} \quad \xi_{n+1} = \iota_n + (-1)^{\iota_n} \left\{ \frac{\xi_n}{\alpha_n} + \Delta_n \right\}_{e_n}.$$

Let $x = x_0 x_1 x_2 \cdots$. We show that $x \in M$ and $\nu(x) = \xi$ by the following steps.

Note. *Consider the case $\beta = 0$. Then for all $n \in \mathbb{N}_0$ we have $\alpha_n = G^n(\alpha)$, $\beta_n = \iota_n = 0$: recall Section 1. So $\Delta_n = e_n = 0$. Hence the definition of x_n and ξ_{n+1} in the case $\beta = 0$ is $x_n = \lfloor \frac{\xi_n}{\alpha_n} \rfloor$ and $\xi_{n+1} = \{ \frac{\xi_n}{\alpha_n} \}$, that is, x is the dual Ostrowski expansion of ξ based on α. Thus Proposition 4.1 is a generalization of dual Ostrowski expansion.*

Step 1: $e_n = 0 \Longrightarrow 0 \le \xi_n < 1$; $e_n = 1 \Longrightarrow 0 < \xi_n \le 1$

Indeed, the case $n = 0$ is clear (recall $e_0 = 0$). Note $e_{n+1} = 0$ if and only if $e_n = \iota_n$.

Step 2: $x_n \in \{0, 1, \cdots, a_n\}$.

Indeed by Step 1

$$e_n = 0 \Longrightarrow \tfrac{\xi_n}{\alpha_n} + \Delta_n \in [\Delta_n, \ \tfrac{1}{\alpha_n} + \Delta_n); \quad e_n = 1 \Longrightarrow \tfrac{\xi_n}{\alpha_n} + \Delta_n \in (\Delta_n, \ \tfrac{1}{\alpha_n} + \Delta_n].$$

By Lemma 2.4 and definitions of a_n and ι_n

$$\frac{1}{\alpha_n} + \Delta_n = \left\lfloor \frac{1}{\alpha_n} \right\rfloor + 1 - \left\{ \frac{-1}{\alpha_n} \right\} + \left\{ \frac{-\beta_n}{\alpha_n} \right\} = a_n + 1 - \left\{ \frac{\beta_n - 1}{\alpha_n} \right\}.$$

So $e_n = 0 \Longrightarrow \frac{\xi_n}{\alpha_n} + \Delta_n \in [0,\ a_n+1)$; $e_n = 1 \Longrightarrow \frac{\xi_n}{\alpha_n} + \Delta_n \in (0,\ a_n+1]$. Hence $0 \le x_n \le a_n$.

Here note that

$$(\dagger) \quad \frac{\xi_n}{\alpha_n} = x_n - (-1)^{\iota_n}\beta_{n+1} + (-1)^{\iota_n}\xi_{n+1}$$

because $\frac{\xi_n}{\alpha_n} + \Delta_n = x_n + \left\{\frac{\xi_n}{\alpha_n} + \Delta_n\right\}_{e_n}$ and $\Delta_n = \iota_n + (-1)^{\iota_n}\beta_{n+1}$ by Remark 2.2.

Step 3: $x_n = 0 \Longrightarrow x_{n+1} \ge_{\iota_n} b_{n+1} - \iota_n$; $x_n = a_n \Longrightarrow x_{n+1} \le_{\iota_n} b_{n+1} + \iota_n$.
Indeed, note that by ($\dagger$) and the definition of b_{n+1}

$$\frac{(-1)^{\iota_n}}{\alpha_{n+1}}\left(\frac{\xi_n}{\alpha_n} - x_n\right) = \frac{\xi_{n+1}}{\alpha_{n+1}} - \frac{\beta_{n+1}}{\alpha_{n+1}} = \frac{\xi_{n+1}}{\alpha_{n+1}} + \Delta_{n+1} - b_{n+1}$$

and so

$$\left[\frac{(-1)^{\iota_n}}{\alpha_{n+1}}\left(\frac{\xi_n}{\alpha_n} - x_n\right)\right]_{e_{n+1}} = x_{n+1} - b_{n+1}.$$

Case 1: $x_n = 0$.
Then

$$x_{n+1} - b_{n+1} = \left[\frac{(-1)^{\iota_n}\xi_n}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}}.$$

If $\iota_n = 0$, then $e_{n+1} = e_n$ and by Step 1

$$\xi_n \begin{cases} \ge 0 \text{ if } e_{n+1} = 0 \\ > 0 \text{ if } e_{n+1} = 1 \end{cases} \text{ and so we have } x_{n+1} - b_{n+1} = \left[\tfrac{\xi_n}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}} \ge 0.$$

If $\iota_n = 1$, then $e_{n+1} = \overline{e_n}$ and by Step 1

$$-\xi_n \begin{cases} < 0 \text{ if } e_{n+1} = 0 \\ \le 0 \text{ if } e_{n+1} = 1 \end{cases} \text{ and so we have } x_{n+1} - b_{n+1} = \left[\tfrac{-\xi_n}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}} \le -1.$$

Hence $x_n = 0 \Longrightarrow x_{n+1} \ge_{\iota_n} b_{n+1} - \iota_n$.
Case 2: $x_n = a_n$.
Then by recursive equation (1) we have $\frac{\xi_n}{\alpha_n} - x_n = \frac{\xi_n - 1}{\alpha_n} + (-1)^{\iota_n}\alpha_{n+1}$ and so

$$x_{n+1} - b_{n+1} = \left[\frac{(-1)^{\iota_n}(\xi_n - 1)}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}} + 1.$$

If $\iota_n = 0$, then $e_{n+1} = e_n$ and by Step 1

$$\xi_n - 1 \begin{cases} < 0 \text{ if } e_{n+1} = 0 \\ \le 0 \text{ if } e_{n+1} = 1 \end{cases} \text{ and so we have } x_{n+1} - b_{n+1} = \left[\tfrac{\xi_n - 1}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}} + 1 \le 0.$$

If $\iota_n = 1$, then $e_{n+1} = \overline{e_n}$ and by Step 1

$$1 - \xi_n \begin{cases} \ge 0 \text{ if } e_{n+1} = 0 \\ > 0 \text{ if } e_{n+1} = 1 \end{cases} \text{ and so we have } x_{n+1} - b_{n+1} = \left[\tfrac{1 - \xi_n}{\alpha_{n+1}\alpha_n}\right]_{e_{n+1}} + 1 \ge 1.$$

Hence $x_n = a_n \Longrightarrow x_{n+1} \le_{\iota_n} b_{n+1} + \iota_n$.

Therefore by Steps 2 and 3, the sequence $x = x_0x_1x_2\cdots$ belongs to M.

Step 4: $\xi = \nu(x)$.
First we claim that for each $N \in \mathbb{N}_0$

$$(*_N) \quad \xi = \sum_{n=0}^{N} \nu_n(x_n) + (-1)^{e_{N+1}}\xi_{N+1}\prod_{j=0}^{N}\alpha_j$$

by induction on N. Indeed by (†)

$$\xi = \xi_0 = (x_0 - (-1)^{\iota_0}\beta_1)\alpha_0 + (-1)^{\iota_0}\xi_1\alpha_0 = \nu_0(x_0) + (-1)^{e_1}\xi_1\alpha_0$$

because $e_0 = 0$ and $e_1 = \iota_0$. So $(*_0)$ holds. Let $N \in \mathbb{N}$ and suppose $(*_{N-1})$ holds, that is,

$$\xi = \sum_{n=0}^{N-1} \nu_n(x_n) + (-1)^{e_N}\xi_N\prod_{j=0}^{N-1}\alpha_j.$$

Since $\xi_N = (x_N - (-1)^{\iota_N}\beta_{N+1})\alpha_N + (-1)^{\iota_N}\xi_{N+1}\alpha_N$ by (†), $(*_N)$ holds (recall $(-1)^{e_{N+1}} = (-1)^{e_N}(-1)^{\iota_N}$). Now by this claim and Lemma 2.6, we have $\xi = \nu(x)$. □

§ 5. Tail inequality

In this section, we show the following two propositions.

Proposition 5.1. *Let $k \in \mathbb{N}_0$, z be k-left extremal and $\widetilde{z}$ be k-right extremal. Then for any $x \in M$ and $l \geq k$,*

$$\sum_{n=k}^{l} \nu_n(z_n) - \prod_{j=0}^{l}\alpha_j \leq \sum_{n=k}^{l}\nu_n(x_n) \leq \sum_{n=k}^{l}\nu_n(\widetilde{z}_n) + \prod_{j=0}^{l}\alpha_j.$$

Hence by Lemmas 2.6, 3.5 and Proposition 5.1, we see that for any $x \in M$, the sequence $\{\sum_{j=0}^{n}\nu_j(x_j)\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence and $\nu(x)$ converges in $[0,1]$.

Proposition 5.2 (Tail inequality). *For any $x \in M$ and $k \in \mathbb{N}_0$*

$$-e_k\prod_{j=0}^{k-1}\alpha_j \leq \sum_{n=k}^{\infty}\nu_n(x_n) \leq \overline{e_k}\prod_{j=0}^{k-1}\alpha_j.$$

We call this inequality **tail inequality**. *Moreover we have the following.*

(1) $\displaystyle\sum_{n=k}^{\infty}\nu_n(x_n) = -e_k\prod_{j=0}^{k-1}\alpha_j$ *if and only if x is k-left extremal.*

(2) $\displaystyle\sum_{n=k}^{\infty}\nu_n(x_n) = \overline{e_k}\prod_{j=0}^{k-1}\alpha_j$ *if and only if x is k-right extremal.*

Note. We will prove local version of tail inequality : see Proposition 8.3 in Section 8.

To prove propositions, we begin with a technical lemma:

Lemma 5.3. *Let $n \in \mathbb{N}_0$, $x \in \mathbb{N}$, $y \in \mathbb{Z}$ with $y \geq -a_n$.*
If $x + y\alpha_n < 0$, then $x = 1$, $y = -a_n$, $\iota_n = 1$ and $x + y\alpha_n = -\alpha_{n+1}\alpha_n$.

Proof. By recursive equation (1)

$$0 > x + y\alpha_n = (x-1) + (y+a_n)\alpha_n + (-1)^{\iota_n}\alpha_{n+1}\alpha_n \geq (-1)^{\iota_n}\alpha_{n+1}\alpha_n$$

hence $\iota_n = 1$ and $x + y\alpha_n = (x-1) + (y+a_n)\alpha_n - \alpha_{n+1}\alpha_n < 0$. Furthermore we see $x = 1$ and $y = -a_n$, because $\alpha_n, \alpha_{n+1} \in (0,1)$. □

From now on we fix $k \in \mathbb{N}_0$.

Let z be k-left extremal and $x \in M$. Define a sequence $m_k m_{k+1} m_{k+2} \cdots$ by

$$m_n = (-1)^{e_n}(x_n - z_n).$$

Then for each $l \geq k$

$$\sum_{n=k}^{l} \nu_n(x_n) - \sum_{n=k}^{l} \nu_n(z_n) = \sum_{n=k}^{l} m_n \prod_{j=0}^{n} \alpha_j.$$

Claim 5.4. For each $n \geq k$ with $n \equiv k \bmod 2$, we have the following.

(1) $m_n \geq 0$.

If $m_n = 0$ then $m_{n+1} \geq 0$.

(2) $m_{n+1} \geq -a_{n+1}$.

If $m_{n+1} = -a_{n+1}$ and $\iota_{n+1} = 1$, then

$$\iota_n = 1,\ b_{n+1} = \begin{cases} 1 & \text{if } e_n = 0 \\ a_{n+1} - 1 & \text{if } e_n = 1 \end{cases} \text{ and } m_{n+2} - 1 \geq \begin{cases} b_{n+2} & \text{if } e_n = 0 \\ a_{n+2} - b_{n+2} & \text{if } e_n = 1. \end{cases}$$

Proof. (1) By definition

$$m_n = \begin{cases} x_n & \text{if } e_n = 0 \\ a_n - x_n & \text{if } e_n = 1 \end{cases}$$

and so $m_n \geq 0$. If $m_n = 0$ (i.e. $x_n = e_n a_n$), then by remark 3.2, $x_{n+1} \geq_{e_{n+1}} z_{n+1}$ thus $m_{n+1} \geq 0$.

(2) By definition

$$m_{n+1} = (-1)^{e_{n+1}}(x_{n+1} - b_{n+1} + (-1)^{e_n}\iota_n) = (-1)^{e_{n+1}}(x_{n+1} - b_{n+1}) - \iota_n.$$

First, we show that $m_{n+1} \geq -a_{n+1}$ and that if $m_{n+1} = -a_{n+1}$, then $x_{n+1} = e_{n+1}a_{n+1}$ and

$$(\diamondsuit) \quad b_{n+1} = \begin{cases} a_{n+1} - \iota_n & \text{if } e_{n+1} = 0 \\ \iota_n & \text{if } e_{n+1} = 1. \end{cases}$$

Case 1: $e_{n+1} = 0$.

By Proposition 2.7, we have

$$m_{n+1} = x_{n+1} - b_{n+1} - \iota_n \geq -b_{n+1} - \iota_n \geq -a_{n+1}.$$

Moreover if $m_{n+1} = -a_{n+1}$, then $x_{n+1} = 0$ and $b_{n+1} = a_{n+1} - \iota_n$.

Case 2: $e_{n+1} = 1$.

By Proposition 2.7, we have

$$m_{n+1} = -x_{n+1} + b_{n+1} - \iota_n \geq -a_{n+1} + b_{n+1} - \iota_n \geq -a_{n+1}.$$

Moreover if $m_{n+1} = -a_{n+1}$, then $x_{n+1} = a_{n+1}$ and $b_{n+1} = \iota_n$.

Next, suppose $m_{n+1} = -a_{n+1}$ and $\iota_{n+1} = 1$. Since $\iota_{n+1} = 1$, we have $b_{n+1} \notin \{0, a_{n+1}\}$ by Remark 2.8. Hence $\iota_n = 1$ by ($\diamondsuit$), and so $e_n = \overline{e_{n+1}} = e_{n+2}$. Moreover since $x \in M$ and $x_{n+1} = e_{n+1}a_{n+1}$, we have $x_{n+2} \geq_{e_n} b_{n+2} + (-1)^{e_n}$ by Remark 3.2. Therefore if $e_n = 0$ then $m_{n+2} = x_{n+2} \geq b_{n+2} + 1$; if $e_n = 1$ then $a_{n+2} - m_{n+2} = x_{n+2} \leq b_{n+2} - 1$. $\square$

Claim 5.5. Let $K \geq k$ be $K \equiv k \bmod 2$. For each $L \in \mathbb{N}$, the following proposition (P_L) holds:

(P_L) If $\displaystyle\sum_{n=K}^{K+2l-1} m_n \prod_{j=0}^{n} \alpha_j < 0$ for each $1 \leq l \leq L$, then

$$(i)\ \iota_n = 1 \quad (K \leq \forall n \leq K + 2L - 1)$$

$$(ii)\ b_n = \begin{cases} 1 & \text{if } e_K = 0 \\ a_n - 1 & \text{if } e_K = 1 \end{cases} \quad (K + 1 \leq \forall n \leq K + 2L - 1)$$

$$(iii)\ \sum_{n=K}^{K+2l-1} m_n \prod_{j=0}^{n} \alpha_j = -\prod_{j=0}^{K+2l} \alpha_j \quad (1 \leq \forall l \leq L)$$

$$(iv)\ m_{K+2L} - 1 \geq \begin{cases} b_{K+2L} & \text{if } e_K = 0 \\ a_{K+2L} - b_{K+2L} & \text{if } e_K = 1. \end{cases}$$

Proof. Let $S_l = \displaystyle\sum_{n=K}^{K+2l-1} m_n \prod_{j=0}^{n} \alpha_j$. We use induction on L.

We show that (P_1) holds. Suppose $S_1 < 0$. Then $m_K + m_{K+1}\alpha_{K+1} < 0$ and so by Claim 5.4 (1), $m_K \geq 1$. Hence by Lemma 5.3, we have $m_{K+1} = -a_{K+1}$, $\iota_{K+1} = 1$ and $S_1 = -\prod_{j=0}^{K+2} \alpha_j$. By Claim 5.4 (2),

$$\iota_K = 1,\ b_{K+1} = \begin{cases} 1 & \text{if } e_K = 0 \\ a_{K+1} - 1 & \text{if } e_K = 1 \end{cases} \text{ and } m_{K+2} - 1 \geq \begin{cases} b_{K+2} & \text{if } e_K = 0 \\ a_{K+2} - b_{K+2} & \text{if } e_K = 1. \end{cases}$$

Thus (P_1) holds.

We show $(P_L) \Longrightarrow (P_{L+1})$. Suppose (P_L) holds and $S_l < 0$ for each $1 \leq l \leq L+1$. It suffices to show the following:

$$\iota_n = 1, \quad b_n = \begin{cases} 1 & \text{if } e_K = 0 \\ a_n - 1 & \text{if } e_K = 1 \end{cases} \quad \text{for } n = K + 2L,\ K + 2L + 1$$

$$S_{L+1} = -\prod_{j=0}^{K+2L+2} \alpha_j$$

$$\text{and } m_{K+2L+2} - 1 \geq \begin{cases} b_{K+2L+2} & \text{if } e_K = 0 \\ a_{K+2L+2} - b_{K+2L+2} & \text{if } e_K = 1. \end{cases}$$

Note $e_{K+2L} = e_{K+2L-2} = \cdots = e_{K+2} = e_K$ by (i) in (P_L). Since $S_L = -\prod_{j=0}^{K+2L} \alpha_j$ by (iii) in (P_L), we have

$$S_{L+1} = (m_{K+2L} - 1 + m_{K+2L+1}\alpha_{K+2L+1}) \prod_{j=0}^{K+2L} \alpha_j.$$

Since $\iota_{K+2L-1} = 1$ by (i) in (P_L), we have by Proposition 2.7

$$1 \le b_{K+2L} \le a_{K+2L} - 1$$

Hence $m_{K+2L} - 1 \ge 1$ by (iv) in (P_L). So since $S_{L+1} < 0$, we have by Lemma 5.3

$$m_{K+2L} - 1 = 1,\ m_{K+2L+1} = -a_{K+2L+1},\ \iota_{K+2L+1} = 1 \text{ and } S_{L+1} = -\prod_{j=0}^{K+2L+2} \alpha_j.$$

The equality $m_{K+2L} - 1 = 1$ implies

$$b_{K+2L} = \begin{cases} 1 & \text{if } e_K = 0 \\ a_{K+2L} - 1 & \text{if } e_K = 1. \end{cases}$$

By Claim 5.4 (2), the equalities $m_{K+2L+1} = -a_{K+2L+1},\ \iota_{K+2L+1} = 1$ and $e_{K+2L} = e_K$ imply

$$\iota_{K+2L} = 1,\ b_{K+2L+1} = \begin{cases} 1 & \text{if } e_K = 0 \\ a_{K+2L+1} - 1 & \text{if } e_K = 1 \end{cases}$$

and

$$m_{K+2L+2} - 1 \ge \begin{cases} b_{K+2L+2} & \text{if } e_K = 0 \\ a_{K+2L+2} - b_{K+2L+2} & \text{if } e_K = 1. \end{cases}$$

Therefore (P_{L+1}) holds. $\square$

Let $\widetilde{z}$ be k-right extremal and $x \in M$. Define a sequence $\widetilde{m_k}\widetilde{m_{k+1}}\widetilde{m_{k+2}}\cdots$ by

$$\widetilde{m_n} = (-1)^{e_n}(\widetilde{z_n} - x_n).$$

Then for each $l \ge k$

$$\sum_{n=k}^{l} \nu_n(\widetilde{z_n}) - \sum_{n=k}^{l} \nu_n(x_n) = \sum_{n=k}^{l} \widetilde{m_n} \prod_{j=0}^{n} \alpha_j.$$

In the same way as the proofs of Claims 5.4 and 5.5, we obtain the following statements:

Claim 5.6. For each $n \ge k$ with $n \equiv k$ mod 2, we have the following.
(1) $\widetilde{m_n} \ge 0$.
If $\widetilde{m_n} = 0$ then $\widetilde{m_{n+1}} \ge 0$.
(2) $\widetilde{m_{n+1}} \ge -a_{n+1}$.
If $\widetilde{m_{n+1}} = -a_{n+1}$ and $\iota_{n+1} = 1$, then

$$\iota_n = 1,\ b_{n+1} = \begin{cases} 1 & \text{if } e_n = 1 \\ a_{n+1} - 1 & \text{if } e_n = 0 \end{cases} \text{ and } \widetilde{m_{n+2}} - 1 \ge \begin{cases} b_{n+2} & \text{if } e_n = 1 \\ a_{n+2} - b_{n+2} & \text{if } e_n = 0. \end{cases}$$

Claim 5.7. Let $K \ge k$ be $K \equiv k$ mod 2. For each $L \in \mathbb{N}$, the following proposition $(\widetilde{P_L})$ holds:

$(\widetilde{P_L})$ If $\displaystyle\sum_{n=K}^{K+2l-1} \widetilde{m_n} \prod_{j=0}^{n} \alpha_j < 0$ for each $1 \leq l \leq L$, then

$$(i)\ \iota_n = 1 \quad (K \leq \forall n \leq K+2L-1)$$

$$(ii)\ b_n = \begin{cases} 1 & \text{if } e_K = 1 \\ a_n - 1 & \text{if } e_K = 0 \end{cases} \quad (K+1 \leq \forall n \leq K+2L-1)$$

$$(iii)\ \sum_{n=K}^{K+2l-1} \widetilde{m_n} \prod_{j=0}^{n} \alpha_j = -\prod_{j=0}^{K+2l} \alpha_j \quad (1 \leq \forall l \leq L)$$

$$(iv)\ \widetilde{m_{K+2L}} - 1 \geq \begin{cases} b_{K+2L} & \text{if } e_K = 1 \\ a_{K+2L} - b_{K+2L} & \text{if } e_K = 0. \end{cases}$$

(**Proof of Proposition** 5.1)
Let $k \in \mathbb{N}_0$, z be k-left extremal, $\widetilde{z}$ be k-right extremal and $x \in M$.
Recall the sequence $m_k m_{k+1} m_{k+2} \cdots$, that is, $m_n = (-1)^{e_n}(x_n - z_n)$, and so for each $l \geq k$

$$\sum_{n=k}^{l} m_n \prod_{j=0}^{n} \alpha_j = \sum_{n=k}^{l} \nu_n(x_n) - \sum_{n=k}^{l} \nu_n(z_n).$$

We show for any $l \geq k$, $\displaystyle\sum_{n=k}^{l} \nu_n(z_n) - \prod_{j=0}^{l} \alpha_j \leq \sum_{n=k}^{l} \nu_n(x_n)$, in other words,

$$(*)_l \quad T_l := \sum_{n=k}^{l} m_n \prod_{j=0}^{n} \alpha_j \geq -\prod_{j=0}^{l} \alpha_j.$$

The inequality $(*)_k$ is clearly holds because $m_k \geq 0$ by Claim 5.4 (1). Let $l > k$. Define

$$J = \left\lfloor \frac{l-k+1}{2} \right\rfloor \geq 1.$$

Then $l \in \{k+2J-1, k+2J\}$ and so $T_l \geq T_{k+2J-1}$ because $m_{k+2J} \geq 0$ by claim 5.4 (1). Hence, in order prove the inequality $(*)_l$, it suffices to show

$$T_{k+2J-1} \geq -\prod_{j=0}^{l} \alpha_j.$$

It suffices to consider the case $T_{k+2J-1} < 0$. Define

$$J_0 = \min\{1 \leq i \leq J \mid i \leq \forall p \leq J,\ T_{k+2p-1} < 0\}.$$

Since $T_{k+2J_0-3} \geq 0$ (if $J_0 \geq 2$), we have

$$\sum_{n=k+2J_0-2}^{k+2p-1} m_n \prod_{j=0}^{n} \alpha_j < 0 \quad \text{for each } J_0 \leq p \leq J.$$

By Claim 5.5 (iii)

$$\sum_{n=k+2J_0-2}^{k+2J-1} m_n \prod_{j=0}^{n} \alpha_j = -\prod_{j=0}^{k+2J} \alpha_j.$$

Therefore

$$T_{k+2J-1} = T_{k+2J_0-3} + \sum_{n=k+2J_0-2}^{k+2J-1} m_n \prod_{j=0}^{n} \alpha_j \geq -\prod_{j=0}^{k+2J} \alpha_j \geq -\prod_{j=0}^{l} \alpha_j$$

(recall $k+2J \geq l$).

Similarly we can show that for any $l \geq k$,

$$\sum_{n=k}^{l} \nu_n(x_n) \leq \sum_{n=k}^{l} \nu_n(\widetilde{z_n}) + \prod_{j=0}^{l} \alpha_j.$$

□

(**Proof of Proposition** 5.2)
Let $k \in \mathbb{N}_0$ and $x \in M$. By Lemmas 2.6, 3.5 and Proposition 5.1, we have the tail inequality:

$$-e_k \prod_{j=0}^{k-1} \alpha_j \leq \sum_{n=k}^{\infty} \nu_n(x_n) \leq \overline{e_k} \prod_{j=0}^{k-1} \alpha_j.$$

Let z be k-left extremal. Recall that for each $l \geq k$, $m_n = (-1)^{e_n}(x_n - z_n)$ and

$$\sum_{n=k}^{l} m_n \prod_{j=0}^{n} \alpha_j = \sum_{n=k}^{l} \nu_n(x_n) - \sum_{n=k}^{l} \nu_n(z_n).$$

By Lemma 3.5, $\sum_{n=k}^{\infty} m_n \prod_{j=0}^{n} \alpha_j = \sum_{n=k}^{\infty} \nu_n(x_n) + e_k \prod_{j=0}^{k-1} \alpha_j$. Hence, in order prove (1) in Proposition 5.2, it suffices to show if $\sum_{n=k}^{\infty} m_n \prod_{j=0}^{n} \alpha_j = 0$ then $m_n = 0$ for each $n \geq k$. To this end, we show a claim:

If there is $r \geq k$ with $r \equiv k \bmod 2$ and $m_r + m_{r+1}\alpha_{r+1} < 0$, then $\sum_{n=k}^{\infty} m_n \prod_{j=0}^{n} \alpha_j > 0$.
Let

$$K = \min\{r \geq k \mid r \equiv k \bmod 2 \text{ and } m_r + m_{r+1}\alpha_{r+1} < 0\}.$$

Then (if $K \geq k+2$) $m_l + m_{l+1}\alpha_{l+1} \geq 0$ for each $k \leq l \leq K-2$ with $l \equiv k \bmod 2$, and so

$$\sum_{n=k}^{K-1} m_n \prod_{j=0}^{n} \alpha_j \geq 0.$$

Assume that for any $l \in \mathbb{N}_0$

$$\sum_{n=K}^{K+2l+1} m_n \prod_{j=0}^{n} \alpha_j < 0.$$

Then by Claim 5.5 (i) and (ii), we have

$$\iota_n = 1 \ (\forall n \geq K) \quad \text{and} \quad b_n = \begin{cases} 1 & \text{if } e_K = 0 \\ a_n - 1 & \text{if } e_K = 1. \end{cases} \ (\forall n \geq K+1)$$

contradicting Proposition 2.7. Hence $\displaystyle\sum_{n=K}^{K+2l+1} m_n \prod_{j=0}^{n} \alpha_j \geq 0$ for some $l \in \mathbb{N}_0$. Let

$$L = \min\{l \in \mathbb{N}_0 \mid \sum_{n=K}^{K+2l+1} m_n \prod_{j=0}^{n} \alpha_j \geq 0\}.$$

Since $m_K + m_{K+1}\alpha_{K+1} < 0$, we see that $L \geq 1$ and for each $1 \leq l \leq L$

$$\sum_{n=K}^{K+2l-1} m_n \prod_{j=0}^{n} \alpha_j < 0.$$

Hence by Claim 5.5 (iii) and (i), we have

$$\sum_{n=K}^{K+2L+1} m_n \prod_{j=0}^{n} \alpha_j = (-1 + m_{K+2L} + m_{K+2L+1}\alpha_{K+2L+1}) \prod_{j=0}^{K+2L} \alpha_j$$

and $\iota_{K+2L-1} = 1$. So $1 \leq b_{K+2L} \leq a_{K+2L} - 1$ by Proposition 2.7, and hence by Claim 5.5 (iv) $-1 + m_{K+2L} \geq 1$. Therefore $\displaystyle\sum_{n=K}^{K+2L+1} m_n \prod_{j=0}^{n} \alpha_j \neq 0$, because α_{K+2L+1} is irrational. Thus

$$\sum_{n=K}^{K+2L+1} m_n \prod_{j=0}^{n} \alpha_j > 0.$$

On the other hand, by Lemma 3.5 and tail inequality, we have

$$\sum_{n=K+2L+2}^{\infty} m_n \prod_{j=0}^{n} \alpha_j = \sum_{n=K+2L+2}^{\infty} \nu_n(x_n) - \sum_{n=K+2L+2}^{\infty} \nu_n(z_n) \geq 0$$

because z is also $(K + 2L + 2)$-left extremal. Summarizing the above, we have

$$\sum_{n=k}^{\infty} m_n \prod_{j=0}^{n} \alpha_j > 0$$

hence the above claim is proved.

Now we prove (1). Suppose $\sum_{n=k}^{\infty} m_n \prod_{j=0}^{n} \alpha_j = 0$. Then $m_i + m_{i+1}\alpha_{i+1} \geq 0$ for each $i \geq k$ with $i \equiv k \bmod 2$ by the above claim. Moreover $m_i + m_{i+1}\alpha_{i+1} = 0$ for each $i \geq k$ with $i \equiv k \bmod 2$. So $m_i = m_{i+1} = 0$ for each $i \geq k$ with $i \equiv k \bmod 2$, because α_{i+1} is irrational. Thus $m_n = 0$ for each $n \geq k$. Similarly we can prove (2). □

§ 6. Doubleton lemma

In preceeding sections, we have constructed the (α, β)-numeration system $\nu : M \to [0, 1]$. Define

$$\{\nu\} : M \to [0, 1) \text{ by } \{\nu\}(x) = \{\nu(x)\} \text{ (the fractional part of } \nu(x)).$$

To show $\{\nu\} : M \to [0, 1)$ is at most 2-to-1, we begin with the following lemma:

Lemma 6.1.

(1) *Let x be k-left extremal and y be $(k-1)$-left extremal.*
If $x_k = y_k$, then $x_n = y_n = e_k a_n$ $(\forall n \geq k)$.

(2) *Let x be k-right extremal and y be $(k-1)$-right extremal.*
If $x_k = y_k$, then $x_n = y_n = \overline{e_k} a_n$ $(\forall n \geq k)$.

(3) *Any left (resp. right) extremal sequence is not right (resp. left) extremal.*

Proof. Note that

$$-(-1)^{e_{k-1}} \iota_{k-1} = (-1)^{e_k} \iota_{k-1}$$

because $(-1)^{e_k} = (-1)^{e_{k-1}}(-1)^{\iota_{k-1}}$ and $(-1)^s s = -s$ for each $s \in \{0,1\}$.

We show (1) and (2). It suffices to show (i) and (ii): for $k \in \mathbb{N}_0$ with $k \geq 1$,
(i) If $e_k a_k = b_k - (-1)^{e_{k-1}} \iota_{k-1}$, then $\iota_n = 0$ $(\forall n \geq k-1)$ and $e_n = e_k,\ b_n = e_k a_n$ $(\forall n \geq k)$.
(ii) If $\overline{e_k} a_k = b_k - (-1)^{\overline{e_{k-1}}} \iota_{k-1}$, then $\iota_n = 0$ $(\forall n \geq k-1)$ and $e_n = e_k,\ b_n = \overline{e_k} a_n$ $(\forall n \geq k)$.
Let $e_k a_k = b_k + (-1)^{e_k} \iota_{k-1}$. So if $e_k = 0$ then $0 = b_k + \iota_{k-1}$; if $e_k = 1$ then $a_k = b_k - \iota_{k-1}$. Hence $\iota_{k-1} = 0$ and $b_k = e_k a_k$. By Remark 2.8, (i) holds. The proof of (ii) is similar. We show (3) by contradiction. Assume there is a sequence z which is l-left and r-right extremal. Then by definition, $r \equiv l + 1 \bmod 2$. Letting $k = \max\{l, r\} + 1$, we have the following system of equations:

$$\begin{array}{l} e_n a_n = z_n = b_n - (-1)^{e_n} \iota_{n-1} \text{ if } n \equiv l \bmod 2 \\ \overline{e_n} a_n = z_n = b_n + (-1)^{e_n} \iota_{n-1} \text{ if } n \equiv l+1 \bmod 2 \end{array} \quad \text{(for each } n \geq k).$$

Case 1: $\exists K \geq k$ such that $\iota_{K-1} = 0$.
Then $b_K \in \{0, a_K\}$. By Remark 2.8, for each $n \geq K$, we have that $\iota_n = 0$ and $e_n = e_K$ and that if $b_K = 0$ then $b_n = 0$; if $b_K = a_K$ then $b_n = a_n$. It contradicts the above system of equations.
Case 2: $\forall n \geq k,\ \iota_{n-1} = 1$.
Then $b_k \in \{1, a_k - 1\}$ and $e_{n+2} = \overline{e_{n+1}} = e_n$ for each $n \geq k$. By the above system of equations, we can see if $b_k = 1$ then $b_n = 1$; if $b_k = a_k - 1$ then $b_n = a_n - 1$. It contradicts Proposition 2.7. □

For each left (resp. right) extremal sequence z, define

$$k(z) = \min\{k \in \mathbb{N}_0 \mid z \text{ is } k\text{-left (resp. } k\text{-right) extremal}\}.$$

For each sequence $x = x_0 x_1 x_2 \cdots$ and each $k \in \mathbb{N}_0$, define

$$x[0,k] = x_0 x_1 \cdots x_k.$$

Now we introduce the main notion of this section:

Definition 6.2 (Doubleton). Let $x \in M$ be left extremal and $y \in M$ be right extremal. We say x and y form a doubleton if the following conditions hold:

$$\begin{array}{rl} (i) & k(x) = k(y) =: k \\ (ii) & x_{k-1} = y_{k-1} + (-1)^{e_{k-1}} \text{ if } k \geq 1 \\ (iii) & x[0, k-2] = y[0, k-2] \text{ if } k \geq 2 \end{array}$$

Lemma 6.3 (Doubleton lemma)**.** *Let $x, y \in M$ with $x \neq y$. Then, we have the following*: $\{\nu\}(x) = \{\nu\}(y)$ *if and only if x and y form a doubleton.*

Proof. Let x and y form a doubleton where x is left extremal and y is right extremal and $k(x) = k(y) = k$. We show $\{\nu\}(x) = \{\nu\}(y)$. In the case $k = 0$, $\nu(x) = 0$ and $\nu(y) = 1$ by Lemma 3.5. Consider the case $k \geq 1$. By Lemma 3.5

$$\begin{aligned}\sum_{n=k-1}^{\infty} \nu_n(x_n) &= \nu_{k-1}(x_{k-1}) - e_k \prod_{j=0}^{k-1} \alpha_j \\ &= \nu_{k-1}(y_{k-1} + (-1)^{e_{k-1}}) - e_k \prod_{j=0}^{k-1} \alpha_j \\ &= \nu_{k-1}(y_{k-1}) + (1 - e_k) \prod_{j=0}^{k-1} \alpha_j \\ &= \sum_{n=k-1}^{\infty} \nu_n(y_n).\end{aligned}$$

Hence $\nu(x) = \nu(y)$.

We show the 'only if' part. Suppose $\{\nu\}(x) = \{\nu\}(y)$.
If $\nu(x) = 0$ (resp. $\nu(x) = 1$), then x is 0-left extremal (resp. 0-right extremal) by Proposition 5.2 and $\nu(y) = 1$ (resp. $\nu(y) = 0$) because $x \neq y$, so x and y form a doubleton. Consider the case $0 < \nu(x) < 1$. Then $\nu(x) = \nu(y)$. Let

$$k = \min\{n \in \mathbb{N}_0 \mid x_n \neq y_n\}.$$

Without the loss of generality, we can suppose $x_k > y_k$. Since $\nu(x) = \nu(y)$,

$$\nu_k(x_k) - \nu_k(y_k) = \sum_{n=k+1}^{\infty} \nu_n(y_n) - \sum_{n=k+1}^{\infty} \nu_n(x_n).$$

Consider the case $e_k = 0$. Then, since

$$\nu_k(x_k) - \nu_k(y_k) = (x_k - y_k) \prod_{j=0}^{k} \alpha_j \geq \prod_{j=0}^{k} \alpha_j$$

and

$$\sum_{n=k+1}^{\infty} \nu_n(y_n) - \sum_{n=k+1}^{\infty} \nu_n(x_n) \leq \overline{e_{k+1}} \prod_{j=0}^{k} \alpha_j - (-e_{k+1}) \prod_{j=0}^{k} \alpha_j = \prod_{j=0}^{k} \alpha_j \text{ (by tail inequality)},$$

we have $x_k = y_k + 1$ and moreover x is $(k+1)$-left extremal and y is $(k+1)$-right extremal by Proposition 5.2. So $k(x) \leq k+1$ by the definition of $k(x)$. We show that $k(x) = k+1$.
Assume that $k(x) < k+1$.
Case 1: $k(x) \equiv k \bmod 2$.
In this case, $x_k = e_k a_k = 0$ (since $e_k = 0$), contradicting $x_k > y_k$.

Case 2: $k(x) \equiv k+1 \bmod 2$.
In this case, $x_{k-1} = e_{k-1}a_{k-1}$ and $x_k = b_k - (-1)^{e_{k-1}}\iota_{k-1}$. Since $y_{k-1} = x_{k-1}$ and $e_k = 0$ and $y \in M$, we have $y_k \geq b_k - (-1)^{e_{k-1}}\iota_{k-1} = x_k$ by Remark 3.2, contradicting $x_k > y_k$.
Hence $k(x) = k+1$. Similarly we can show $k(y) = k+1$. So x and y form a doubleton. The proof in the case $e_k = 1$ is also similar. □

Denote by $R_\alpha : [0,1) \to [0,1)$ the rotation by angle α, that is, $R_\alpha(\xi) = \{\xi + \alpha\}$, and by $\mathcal{O}_\xi$ the orbit of ξ under R_α, that is, $\mathcal{O}_\xi = \{R_\alpha^n(\xi) \mid n \in \mathbb{Z}\}$.

Lemma 6.4. *If $x \in M$ is extremal, then $\{\nu\}(x) \in \mathcal{O}_\alpha \cup \mathcal{O}_\beta$.*

Proof. We show that for each $N \in \mathbb{N}_0$

$$(i)\ \sum_{n=0}^{2N} \nu_n(0) = \beta - \sum_{n=0}^{N} (-1)^{e_{2n}}(b_{2n} - \iota_{2n}) \prod_{j=0}^{2n} \alpha_j$$

$$(ii)\ \sum_{n=0}^{2N+1} \nu_n(0) = -\sum_{n=0}^{N} (-1)^{e_{2n+1}}(b_{2n+1} - \iota_{2n+1}) \prod_{j=0}^{2n+1} \alpha_j.$$

Note that for each $n \in \mathbb{N}_0$

$$\begin{aligned}
\nu_n(0) + \nu_{n+1}(0) &= -(-1)^{e_n}(-1)^{\iota_n}\beta_{n+1}\prod_{j=0}^{n}\alpha_j - (-1)^{e_{n+1}}(-1)^{\iota_{n+1}}\beta_{n+2}\prod_{j=0}^{n+1}\alpha_j \\
&= -(-1)^{e_{n+1}}\left(\frac{\beta_{n+1}}{\alpha_{n+1}} + (-1)^{\iota_{n+1}}\beta_{n+2}\right)\prod_{j=0}^{n+1}\alpha_j \\
&= -(-1)^{e_{n+1}}(b_{n+1} - \iota_{n+1})\prod_{j=0}^{n+1}\alpha_j \quad \text{(by recursive equation (2))}.
\end{aligned}$$

When $N = 0$, by recursive equation (2) we have $\nu_0(0) = -(-1)^{\iota_0}\beta_1\alpha_0 = \beta - (b_0 - \iota_0)\alpha_0$. When $N \geq 1$,

$$\sum_{n=0}^{2N}\nu_n(0) = \nu_0(0) + \sum_{n=1}^{N}\Big(\nu_{2n-1}(0) + \nu_{2n}(0)\Big) = \beta - (b_0 - \iota_0)\alpha_0 - \sum_{n=1}^{N}(-1)^{e_{2n}}(b_{2n} - \iota_{2n})\prod_{j=0}^{2n}\alpha_j.$$

So (i) holds. On the other hand

$$\sum_{n=0}^{2N+1}\nu_n(0) = \sum_{n=0}^{N}\Big(\nu_{2n}(0) + \nu_{2n+1}(0)\Big) = -\sum_{n=0}^{N}(-1)^{e_{2n+1}}(b_{2n+1} - \iota_{2n+1})\prod_{j=0}^{2n+1}\alpha_j,$$

that is, (ii) also holds.

Now, let $x \in M$ be k-left extremal. We show $\{\nu\}(x) \in \mathcal{O}_\alpha \cup \mathcal{O}_\beta$. When $k = 0$, we have $\nu(x) = 0 \in \mathcal{O}_\alpha$ by Lemma 3.5. When $k \geq 1$, by Lemma 3.5

$$\nu(x) = \sum_{n=0}^{k-1}\nu_n(x_n) - e_k\prod_{j=0}^{k-1}\alpha_j = \sum_{n=0}^{k-1}\nu_n(0) + \sum_{n=0}^{k-1}(-1)^{e_n}x_n\prod_{j=0}^{n}\alpha_j - e_k\prod_{j=0}^{k-1}\alpha_j$$

and so, by $(i), (ii)$ and Lemma 2.5, we have $\{\nu\}(x) \in \mathcal{O}_\alpha \cup \mathcal{O}_\beta$. In the same way, we can show that $\{\nu\}(x) \in \mathcal{O}_\alpha \cup \mathcal{O}_\beta$ for any right extremal sequence $x \in M$. □

Remark 6.5. *By Lemma 6.3, the map $\{\nu\} : M \to [0,1)$ is at most 2-to-1: more precisely we have (with Lemma 6.4)*

$$\begin{aligned}
&\{\xi \in [0,1) \mid \sharp\{\nu\}^{-1}(\xi) \geq 2\} \\
&\subset \{\xi \in [0,1) \mid \xi = \{\nu\}(x) = \{\nu\}(y) \textit{ for some doubleton } \{x,y\}\} \\
&\subset \{\{\nu\}(x) \mid x \in M : \textit{left extremal}\} \cap \{\{\nu\}(y) \mid y \in M : \textit{right extremal}\} \\
&\subset \{\{\nu\}(x) \mid x \in M : \textit{left extremal}\} \cup \{\{\nu\}(y) \mid y \in M : \textit{right extremal}\} \\
&\subset \mathcal{O}_\alpha \cup \mathcal{O}_\beta.
\end{aligned}$$

§ 7. Odometer on M

In this section, we introduce the odometer $H : M \to M$ and study its properties.

Definition 7.1. Define the sequences c and $a - c$ by

$$c_n = \begin{cases} a_n - \iota_n & \text{if } n \text{ is even} \\ \iota_n & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad (a-c)_n = a_n - c_n.$$

for each $n \in \mathbb{N}_0$. Thus, $c = (a_0 - \iota_0)\iota_1(a_2 - \iota_2)\iota_3 \cdots$ and $a - c = \iota_0(a_1 - \iota_1)\iota_2(a_3 - \iota_3) \cdots$.

Note $a_n - \iota_n = \left\lfloor \frac{1}{\alpha_n} \right\rfloor > 0$. Recall conditions $(1)_n$ and $(2)_n$ in Definition 3.1:

$$\begin{aligned}
&(1)_n \; x_n = 0 \implies x_{n+1} \geq_{\iota_n} b_{n+1} - \iota_n \\
&(2)_n \; x_n = a_n \implies x_{n+1} \leq_{\iota_n} b_{n+1} + \iota_n.
\end{aligned}$$

Lemma 7.2. $\{c,\ a - c\} \subset M$.

Proof. Since $a_n \neq \iota_n$, it suffices to show c (resp. $a - c$) satisfies conditions $(1)_{2n+1}, (2)_{2n}$ (resp. $(1)_{2n}, (2)_{2n+1}$) for each $n \in \mathbb{N}_0$. We can show them by using following claim.

Claim: $\iota_n = 0 \implies \iota_{n+1} \leq b_{n+1} \leq a_{n+1} - \iota_{n+1}$. Indeed, when $\iota_n = 0$, we have, by Proposition 2.7 and Remark 2.8, $0 \leq b_{n+1} \leq a_{n+1}$ and if $\iota_{n+1} = 1$ then $b_{n+1} \notin \{0, a_{n+1}\}$. □

For each sequence $x = x_0x_1x_2 \cdots$ and each $k \in \mathbb{N}_0$, define

$$x[k, \infty) = x_k x_{k+1} x_{k+2} \cdots .$$

Definition 7.3 (Odometer)**.** For each $x \in M$, define a sequence $H(x)$ $(= H_{\alpha,\beta}(x))$ as follows. Define

$$H(c) = a - c.$$

Let $c \neq x \in M$ and define

$$L = L(x) = \min\{n \in \mathbb{N}_0 \mid x_n \neq c_n\}.$$

Case (1) : $L = 0$, or $L > 0$ is even with $x_L \geq b_L$. Define

$$H(x) = \begin{cases} (a-c)[0, L-2](a_{L-1} - \overline{\iota_{L-1}})(x_L + 1)x[L+1, \infty) \\ \qquad\qquad \text{if } x_L < a_L - 1 \text{ or if } x_L = a_L - 1 \text{ and } x_{L+1} \leq b_{L+1} \\ (a-c)[0, L](x_{L+1} - 1)x[L+2, \infty) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Case (2) : $L > 0$ is even with $x_L < b_L$. Define

$$H(x) = (a-c)[0, L-3]\ \overline{\iota_{L-2}}\ 0\ x[L, \infty).$$

Case (3) : L is odd with $x_L \leq b_L$. Define

$$H(x) = \begin{cases} (a-c)[0, L-2]\ \overline{\iota_{L-1}}\ (x_L - 1)x[L+1, \infty) \\ \qquad\qquad \text{if } x_L > 1 \text{ or if } x_L = 1 \text{ and } x_{L+1} \geq b_{L+1} \\ (a-c)[0, L](x_{L+1} + 1)x[L+2, \infty) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Case (4) : L is odd with $x_L > b_L$. Define

$$H(x) = (a-c)[0, L-3](a_{L-2} - \overline{\iota_{L-2}})a_{L-1}x[L, \infty).$$

Note. Consider the case $\beta = 0$. For all $n \in \mathbb{N}_0$, $\iota_n = b_n = 0$ and $M = M^\alpha = \{x \in \prod_{n \in \mathbb{N}_0}\{0, 1, \cdots, a_n\} \mid x_n = a_n \Longrightarrow x_{n+1} = 0\}$: recall Section 1. Hence $c = a_0 0 a_2 0 \cdots$ and the case (1) in Definition 7.3 only occurs. So $H = H_\alpha$ (dual Ostrowski odometer).

Example (continued). Let $\alpha = \sqrt{2} - 1$ and $\beta = 1 - \frac{1}{\sqrt{2}}$. In this example, $\iota_n = 0$, $a_n = 2$ and $b_n = 1$ for each $n \in \mathbb{N}_0$: recall Section 1. So $c = 2020\cdots$ and $a - c = 0202\cdots$. Since $\iota_n = 0$ $(\forall n)$, cases (2) and (4) in Definition 7.3 do not occur by Claim 7.4 (i) (see below). Let $c \neq x \in M$ and $L = L(x)$. When L is even (so $x_L \neq 2$ and if $L > 0$ then $x_L \neq 0$)

$$H(x) = \begin{cases} 0202\cdots0201(x_L + 1)x[L+1, \infty) & \text{if } L = 0 \text{ and } x_0 = 0 \text{ or if } x_L = 1 \text{ and } x_{L+1} \leq 1 \\ 0202\cdots0201x[L+2, \infty) & \text{otherwise (that is, } x_L x_{L+1} = 12), \end{cases}$$

and when L is odd (so $x_L = 1$)

$$H(x) = \begin{cases} 0202\cdots021(x_L - 1)x[L+1, \infty) & \text{if } x_L = 1 \text{ and } x_{L+1} \geq 1 \\ 0202\cdots021x[L+2, \infty) & \text{otherwise (that is, } x_L x_{L+1} = 10). \end{cases}$$

□

In order to show $H(M) \subset M$, we prepare the following technical claim.

Claim 7.4. Let $c \neq x \in M$ and $L = L(x)$.
(i) In case (2) or (4) (i.e. when $L > 0$ is even and $x_L < b_L$ or when L is odd and $x_L > b_L$),

$$\iota_{L-1} = 1,\ H(x)_{L-1} = \begin{cases} x_{L-1} - 1 \text{ if } L \text{ is even} \\ x_{L-1} + 1 \text{ if } L \text{ is odd.} \end{cases} \text{ and } H(x)_{L-2} \begin{cases} \leq b_{L-2} \text{ if } L \text{ is even} \\ \geq b_{L-2} \text{ if } L > 1 \text{ is odd.} \end{cases}$$

(ii) When $L > 0$ is even with $a_L - 1 \geq x_L \geq b_L$ or when L is odd with $1 \leq x_L \leq b_L$,

$$H(x)_{L-1} \begin{cases} \geq b_{L-1} \text{ if } L \text{ is even} \\ \leq b_{L-1} \text{ if } L \text{ is odd.} \end{cases}$$

(iii) When L is even with $x_L = a_L$ or when L is odd with $x_L = 0$,

$$x_{L+1} \begin{cases} > b_{L+1} \text{ if } L \text{ is even} \\ < b_{L+1} \text{ if } L \text{ is odd.} \end{cases}$$

Proof. (i) We show $\iota_{L-1} = 1$ in case (2) or (4). Indeed suppose $L > 0$ and $\iota_{L-1} = 0$. Then, by definitions of $L(x)$ and c,

$$x_{L-1} = c_{L-1} = \begin{cases} 0 & \text{if } L \text{ is even} \\ a_{L-1} & \text{if } L \text{ is odd.} \end{cases}$$

Since x satisfies conditions $(1)_{L-1}$ and $(2)_{L-1}$,

$$x_L \begin{cases} \geq b_L \text{ if } L \text{ is even} \\ \leq b_L \text{ if } L \text{ is odd.} \end{cases}$$

Hence, in case (2) or (4), $\iota_{L-1} = 1$ and $x_{L-1} = 1$ if L is even; $x_{L-1} = a_{L-1} - 1$ if L is odd. So, in these cases, $H(x)_{L-1} = 0 = x_{L-1} - 1$ if L is even; $H(x)_{L-1} = a_{L-1} = x_{L-1} + 1$ if L is odd.

Consider the case (2). Then $b_L \neq 0$ (since $0 \leq x_L < b_L$) and hence $b_{L-2} \geq 1$ by Remark 2.8. So $H(x)_{L-2} = \overline{\iota_{L-2}} \leq b_{L-2}$.

Consider the case (4) with $L > 1$. Then $b_L \neq a_L$ (since $a_L \geq x_L > b_L$) and hence $b_{L-2} \leq a_{L-2} - 1$ by Remark 2.8. So $H(x)_{L-2} = a_{L-2} - \overline{\iota_{L-2}} \geq b_{L-2}$.

(ii) If $L > 0$ is even with $a_L - 1 \geq x_L \geq b_L$ then $H(x)_{L-1} \geq a_{L-1} - 1$ and $a_{L-1} - 1 \geq b_{L-1}$ by Remark 2.8. Similarly, if L is odd with $1 \leq x_L \leq b_L$ then $H(x)_{L-1} \leq 1 \leq b_{L-1}$.

(iii) Suppose L is even with $x_L = a_L$. Since $x_L \neq c_L = a_L - \iota_L$, we have $\iota_L = 1$ and so $x_{L+1} \geq b_{L+1} + 1$ because x satisfies condition $(2)_L$. The proof in case that L is odd with $x_L = 0$ is similar. $\square$

Now we show $H(M) \subset M$; the proof may look somewhat tedious.

Lemma 7.5. *For each $x \in M$, $H(x) \in M$. We call $H : M \to M$ the (α, β)-***odometer**.

Proof. It suffices to consider the case $x \neq c$. Let $L = L(x)$.

Case (1): $L = 0$, or $L > 0$ is even and $x_L \geq b_L$.
Subcase (1)-1: $x_L < a_L - 1$, or $x_L = a_L - 1$ with $x_{L+1} \leq b_{L+1}$.
In this subcase,

$$H(x) = (a - c)[0, L-2](a_{L-1} - \overline{\iota_{L-1}})(x_L + 1)x[L+1, \infty).$$

Note that $(a - c)_{L-2} = \iota_{L-2}$ if $L > 0$. It suffices to show that $H(x)$ satisfies $(2)_L$, and $(1)_{L-1}, (2)_{L-1}, (1)_{L-2}$ if $L > 0$. Indeed suppose $H(x)_L = a_L$. Then $x_L = a_L - 1$ and so we have $x_{L+1} \leq b_{L+1}$ and $\iota_L = 0$ (because $x_L \neq c_L = a_L - \iota_L$). Therefore $H(x)_{L+1} \leq_{\iota_L} b_{L+1} + \iota_L$, that is, $(2)_L$ holds. Suppose $L > 0$. Since $x_L \geq b_L$, we can see that $H(x)$ satisfies $(1)_{L-1}, (2)_{L-1}$ (note $a_{L-1} - \overline{\iota_{L-1}} = 0 \Longrightarrow \iota_{L-1} = 0$, because $a_{L-1} \geq 1$). If $H(x)_{L-2} = 0$, then $\iota_{L-2} = 0$ and so $H(x)_{L-1} \geq_{\iota_{L-2}} b_{L-1} - \iota_{L-2}$ by Claim 7.4 (ii), that is, $(1)_{L-2}$ holds.

Subcase (1)-2: $x_L = a_L - 1$ with $x_{L+1} > b_{L+1}$, or $x_L = a_L$.

In this subcase,

$$H(x) = (a-c)[0, L](x_{L+1} - 1)x[L+2, \infty).$$

Note that $(a-c)_L = \iota_L$ and $x_{L+1} > b_{L+1}$ by Claim 7.4 (iii). It suffices to show that $H(x)$ satisfies $(1)_{L+1}$ and $(1)_L$. Suppose $H(x)_{L+1} = 0$. Then $b_{L+1} = 0$ (since $x_{L+1} - 1 \geq b_{L+1}$). So by Remark 2.8, we have $\iota_{L+1} = 0 = b_{L+2}$. Hence $H(x)_{L+2} \geq_{\iota_{L+1}} 0 = b_{L+2} - \iota_{L+1}$, that is, $(1)_{L+1}$ holds. Since $x_{L+1} - 1 \geq b_{L+1}$, we can see that $H(x)$ satisfies $(1)_L$.

Case (2): $L > 0$ is even and $x_L < b_L$.

In this case,

$$H(x) = (a-c)[0, L-3]\ \overline{\iota_{L-2}}\ 0\ x[L, \infty).$$

Note that $(a-c)_{L-3} = a_{L-3} - \iota_{L-3}$ if $L > 2$. It suffices to show that $H(x)$ satisfies $(1)_{L-1}, (1)_{L-2}, (2)_{L-2}$ and $(2)_{L-3}$ if $L > 2$. Since $\iota_{L-1} = 1$ (by Claim 7.4 (i)) and $x_L < b_L$, we can see that $H(x)$ satisfies $(1)_{L-1}$. Suppose $H(x)_{L-2} = 0$. Then $\iota_{L-2} = 1$ and so $H(x)_{L-1} = 0 \geq_{\iota_{L-2}} b_{L-1} - \iota_{L-2}$ (by Proposition 2.7), that is, $(1)_{L-2}$ holds. We can see that $H(x)$ satisfies $(2)_{L-2}$ (note $\overline{\iota_{L-2}} = a_{L-2} \Longrightarrow \iota_{L-2} = 0$, because $a_{L-2} \geq 1$). Suppose $L > 2$ and $H(x)_{L-3} = a_{L-3}$. Then $\iota_{L-3} = 0$ and so $H(x)_{L-2} \leq_{\iota_{L-3}} b_{L-2} + \iota_{L-3}$ by Claim 7.4 (i), that is, $(2)_{L-3}$ holds.

Case (3): L is odd and $x_L \leq b_L$.

Subcase (3)-1: $x_L > 1$, or $x_L = 1$ with $x_{L+1} \geq b_{L+1}$.

In this subcase,

$$H(x) = (a-c)[0, L-2]\ \overline{\iota_{L-1}}\ (x_L - 1)x[L+1, \infty).$$

We can see that $H(x) \in M$ by the similar argument to Subcase (1)-1.

Subcase (3)-2: $x_L = 1$ with $x_{L+1} < b_{L+1}$, or $x_L = 0$.

In this subcase,

$$H(x) = (a-c)[0, L](x_{L+1} + 1)x[L+2, \infty).$$

We can see that $H(x) \in M$ by the similar argument to Subcase (1)-2.

Case (4): L is odd and $x_L > b_L$.

In this case,

$$H(x) = (a-c)[0, L-3](a_{L-2} - \overline{\iota_{L-2}})a_{L-1}x[L, \infty).$$

We can see that $H(x) \in M$ by the similar argument to Case (2). □

Here we equip the space $\prod_{n \in \mathbb{N}_0} \{0, 1, \cdots, a_n\}$ with a usual metric d defined by $d(x, y) = (1 + \min\{n \in \mathbb{N}_0 \mid x_n \neq y_n\})^{-1}$ for $x \neq y$. Then M is compact and moreover by the definition of $L(x)$, we have

Remark 7.6. *$H : M \to M$ is continuous.*

Now we introduce *carry formula*:

Carry formula

$(C)_0$ $\nu(x) = 1 + \nu((x_0 - a_0)(x_1 - 1)x[2, \infty))$

$(C)_n$ $\nu(x) = \nu\Big(x[0, n-2](x_{n-1} + (-1)^{\iota_{n-1}})(x_n - a_n)(x_{n+1} - 1)x[n+2, \infty)\Big)$ for $n \in \mathbb{N}$

Proof of Carry formula. Recall the definition of $\nu_n(x_n)$ in Definition 3.3. First, by $e_0 = 0,\ e_1 = \iota_0$ and recursive equation (1), we have

$$1 + \nu_0(x_0 - a_0) + \nu_1(x_1 - 1) = 1 + \nu_0(x_0) - a_0\alpha_0 + \nu_1(x_1) - (-1)^{\iota_0}\alpha_1\alpha_0 = \nu_0(x_0) + \nu_1(x_1)$$

and so carry formula $(C)_0$ holds. Let $n \in \mathbb{N}$. By multiplying both sides of recursive equation (1) into $(-1)^{e_n}\prod_{j=0}^{n}\alpha_j$, we have

$$(-1)^{\iota_{n-1}}(-1)^{e_{n-1}}\prod_{j=0}^{n-1}\alpha_j = a_n(-1)^{e_n}\prod_{j=0}^{n}\alpha_j + (-1)^{e_{n+1}}\prod_{j=0}^{n+1}\alpha_j$$

(recall $(-1)^{e_{n+1}} = (-1)^{e_n}(-1)^{\iota_n}$). Hence

$$\nu_{n-1}(x_{n-1} + (-1)^{\iota_{n-1}}) + \nu_n(x_n - a_n) + \nu_{n+1}(x_{n+1} - 1) = \nu_{n-1}(x_{n-1}) + \nu_n(x_n) + \nu_{n+1}(x_{n+1})$$

and so carry formula $(C)_n$ also holds. $\square$

Define $\underset{\nu}{=}$ by

$$x \underset{\nu}{=} y \iff \{\nu\}(x) = \{\nu\}(y).$$

Then we can rewrite carry formula as

$$\begin{aligned}
(C)_0^-\ x &\underset{\nu}{=} (x_0 - a_0)(x_1 - 1)x[2, \infty)\\
(C)_0^+\ x &\underset{\nu}{=} (x_0 + a_0)(x_1 + 1)x[2, \infty)\\
(C)_n^-\ x &\underset{\nu}{=} x[0, n-2](x_{n-1} + (-1)^{\iota_{n-1}})(x_n - a_n)(x_{n+1} - 1)x[n+2, \infty) \quad \text{for } n \in \mathbb{N}\\
(C)_n^+\ x &\underset{\nu}{=} x[0, n-2](x_{n-1} - (-1)^{\iota_{n-1}})(x_n + a_n)(x_{n+1} + 1)x[n+2, \infty) \quad \text{for } n \in \mathbb{N}.
\end{aligned}$$

By using carry formula, we have **carry operation**: Typical operation is as follows. (Note $\overline{s} = s + (-1)^s$ for each $s \in \{0, 1\}$.)

$$\begin{array}{rcccccl}
(c_0+1)c[1,\infty) = & (a_0 + \overline{\iota_0}) & \iota_1 & (a_2 - \iota_2) & \iota_3 & (a_4 - \iota_4) & c[5,\infty) \\
\\
\underset{\nu}{=} & \overline{\iota_0} & (-\overline{\iota_1}) & (a_2 - \iota_2) & \iota_3 & (a_4 - \iota_4) & c[5,\infty) \text{ by } (C)_0^- \\
\underset{\nu}{=} & \iota_0 & (a_1 - \overline{\iota_1}) & (a_2 + \overline{\iota_2}) & \iota_3 & (a_4 - \iota_4) & c[5,\infty) \text{ by } (C)_1^+ \\
\underset{\nu}{=} & \iota_0 & (a_1 - \iota_1) & \overline{\iota_2} & (-\overline{\iota_3}) & (a_4 - \iota_4) & c[5,\infty) \text{ by } (C)_2^- \\
\underset{\nu}{=} & \iota_0 & (a_1 - \iota_1) & \iota_2 & (a_3 - \overline{\iota_3}) & (a_4 + \overline{\iota_4}) & c[5,\infty) \text{ by } (C)_3^+
\end{array}$$

and so on. Now we can show

Lemma 7.7. $\{\nu\} \circ H = R_\alpha \circ \{\nu\}$.

Proof. Note that for each $x \in M$

$$R_\alpha(\{\nu\}(x)) = \{\nu(x) + \alpha\} = \{\nu\}\Big((x_0 + 1)x[1, \infty)\Big).$$

It is sufficient to show $(x_0 + 1)x[1, \infty) \underset{\nu}{=} H(x)$. First we have $(c_0 + 1)c[1, \infty) \underset{\nu}{=} a - c = H(c)$ by using the above carry operation indefinitely.
Next let $c \neq x \in M$ and $L = L(x)$.
Case 1: $L = 0$.
• If $x_0 < a_0 - 1$ or if $x_0 = a_0 - 1$ and $x_1 \leq b_1$, then $(x_0 + 1)x[1, \infty) = H(x)$ by definition.
• If $x_0 = a_0 - 1$ and $x_1 > b_1$ or if $x_0 = a_0$, then $x_0 = a_0 - \overline{\iota_0}$ (because $x_0 \neq c_0 = a_0 - \iota_0$) and so by carry formula $(C)_0^-$

$$(x_0 + 1)x[1, \infty) = (a_0 + \iota_0)x[1, \infty) \underset{\nu}{=} \iota_0(x_1 - 1)x[2, \infty) = (a - c)_0(x_1 - 1)x[2, \infty) = H(x).$$

Case 2: $L \geq 1$.
Then $(x_0 + 1)x[1, \infty) = (c_0 + 1)c[1, L-1]x[L, \infty)$. By carry operation as above (via carry formulas $(C)_0^-, (C)_1^+, \cdots, (C)_{L-2}^{\mp}$), we have

$$(x_0 + 1)x[1, \infty) \underset{\nu}{=} \begin{cases} (a - c)[0, L - 3]\overline{\iota_{L-2}}(-\overline{\iota_{L-1}})x[L, \infty) & \text{if } L \text{ is even} \\ (a - c)[0, L - 3](a_{L-2} - \overline{\iota_{L-2}})(a_{L-1} + \overline{\iota_{L-1}})x[L, \infty) & \text{if } L \text{ is odd.} \end{cases}$$

Subcase 2-1: L is even.
• If $x_L < b_L$ (i.e. Case (2) in Definition 7.3), then $\iota_{L-1} = 1$ by Claim 7.4 (i) and so we have $(x_0 + 1)x[1, \infty) \underset{\nu}{=} (a - c)[0, L - 3]\overline{\iota_{L-2}}\ 0\ x[L, \infty) = H(x)$.
• Suppose $x_L \geq b_L$. By carry formula $(C)_{L-1}^+$

$$\begin{aligned}(x_0 + 1)x[1, \infty) &\underset{\nu}{=} (a - c)[0, L - 3]\iota_{L-2}(a_{L-1} - \overline{\iota_{L-1}})(x_L + 1)x[L + 1, \infty) \\ &= (a - c)[0, L - 2](a_{L-1} - \overline{\iota_{L-1}})(x_L + 1)x[L + 1, \infty).\end{aligned}$$

∗ If $x_L < a_L - 1$ or if $x_L = a_L - 1$ and $x_{L+1} \leq b_{L+1}$, then $(x_0 + 1)x[1, \infty) \underset{\nu}{=} H(x)$.
∗ If $x_L = a_L - 1$ and $x_{L+1} > b_{L+1}$ or if $x_L = a_L$, then $x_L = a_L - \overline{\iota_L}$ (since $x_L \neq c_L = a_L - \iota_L$) and so

$$\begin{aligned}(x_0 + 1)x[1, \infty) &\underset{\nu}{=} (a - c)[0, L - 2](a_{L-1} - \overline{\iota_{L-1}})(a_L + \iota_L)x[L + 1, \infty) \\ &\underset{\nu}{=} (a - c)[0, L - 2](a_{L-1} - \iota_{L-1})\iota_L(x_{L+1} - 1)x[L + 2, \infty) \quad \text{by } (C)_L^- \\ &= (a - c)[0, L](x_{L+1} - 1)x[L + 2, \infty) = H(x).\end{aligned}$$

Subcase 2-2: L is odd.
Similarly we can show $(x_0 + 1)x[1, \infty) \underset{\nu}{=} H(x)$. □

Discussion. In the above proof, we used carry operation. Remenber the outline of this proof: First consider the sequence $(x_0 + 1)x[1, \infty)$ (*naive adding* 1). We applied carry operation to $(x_0 + 1)x[1, \infty)$ in order to make the deformed sequence belong to M (*normalization by carry*), and then the normalized sequence is $H(x)$. In this process, we used carry operation at most

$L(x)+1$ times (precisely, $L(x)-1, L(x)$ or $L(x)+1$ times). Moreover we will see (by theorem 1.1) that, among deformed-by-carry-operation sequences of $(x_0+1)x[1,\infty)$, $H(x)$ is the unique sequence which belongs to M. $\square$

Next we show $H: M \to M$ is a bijection. Before making a formal definition of the inverse H^{-1} we give another carry operation, by using carry formulas $(C)_0^+, (C)_1^-, (C)_2^+$ and $(C)_3^-$, in the following way:

$$\begin{array}{rlllllll}
((a-c)_0-1)(a-c)[1,\infty) = & (-\overline{\iota_0}) & (a_1-\iota_1) & \iota_2 & (a_3-\iota_3) & \iota_4 & (a-c)[5,\infty) \\
\underset{\nu}{=} & (a_0-\overline{\iota_0}) & (a_1+\overline{\iota_1}) & \iota_2 & (a_3-\iota_3) & \iota_4 & (a-c)[5,\infty) \\
\underset{\nu}{=} & c_0 & \overline{\iota_1} & (-\overline{\iota_2}) & (a_3-\iota_3) & \iota_4 & (a-c)[5,\infty) \\
\underset{\nu}{=} & c_0 & c_1 & (a_2-\overline{\iota_2}) & (a_3+\overline{\iota_3}) & \iota_4 & (a-c)[5,\infty) \\
\underset{\nu}{=} & c_0 & c_1 & c_2 & \overline{\iota_3} & (-\overline{\iota_4}) & (a-c)[5,\infty)
\end{array}$$

and so on. This is the inverse operation of *adding* 1 (i.e. H), that is, *adding* (-1).

Definition 7.8. For each $x \in M$, define a sequence $K(x)$ as follows. Define firstly

$$K(a-c)=c.$$

Let $a-c \neq x \in M$ and define

$$J = J(x) = \min\{n \in \mathbb{N}_0 \mid x_n \neq a_n - c_n\}.$$

Case (I): $J=0$, or $J>0$ is even with $x_J \le b_J$. Define

$$K(x) = \begin{cases} c[0,J-2]\ \overline{\iota_{J-1}}\ (x_J-1)x[J+1,\infty) \\ \qquad\qquad \text{if } x_J > 1 \text{ or if } x_J = 1 \text{ and } x_{J+1} \ge b_{J+1} \\ c[0,J](x_{J+1}+1)x[J+2,\infty) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Case (II): $J>0$ is even with $x_J > b_J$. Define

$$K(x) = c[0,J-3](a_{J-2}-\overline{\iota_{J-2}})a_{J-1}x[J,\infty).$$

Case (III): J is odd with $x_J \ge b_J$. Define

$$K(x) = \begin{cases} c[0,J-2](a_{J-1}-\overline{\iota_{J-1}})(x_J+1)x[J+1,\infty) \\ \qquad\qquad \text{if } x_J < a_J - 1 \text{ or if } x_J = a_J - 1 \text{ and } x_{J+1} \le b_{J+1} \\ c[0,J](x_{J+1}-1)x[J+2,\infty) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Case (IV): J is odd with $x_J < b_J$. Define

$$K(x) = c[0,J-3]\ \overline{\iota_{J-2}}\ 0\ x[J,\infty).$$

In the same way as the proofs in Claim 7.4 and Lemma 7.5, we can show the following:

Claim 7.9. Let $a - c \neq x \in M$ and $J = J(x)$.
(i) In case (II) or (IV) (i.e. when $J > 0$ is even with $x_J > b_J$ or J is odd with $x_J < b_J$),

$$\iota_{J-1} = 1,\ K(x)_{J-1} = \begin{cases} x_{J-1} + 1 \text{ if } J \text{ is even} \\ x_{J-1} - 1 \text{ if } J \text{ is odd.} \end{cases} \text{ and } K(x)_{J-2} \begin{cases} \geq b_{J-2} \text{ if } J \text{ is even} \\ \leq b_{J-2} \text{ if } J > 1 \text{ is odd.} \end{cases}$$

(ii) When $J > 0$ is even with $1 \leq x_J \leq b_J$ or J is odd with $a_J - 1 \geq x_J \geq b_J$,

$$K(x)_{J-1} \begin{cases} \leq b_{J-1} \text{ if } J \text{ is even} \\ \geq b_{J-1} \text{ if } J \text{ is odd.} \end{cases}$$

(iii) When J is even with $x_J = 0$ or J is odd with $x_J = a_J$,

$$x_{J+1} \begin{cases} < b_{J+1} \text{ if } J \text{ is even} \\ > b_{J+1} \text{ if } J \text{ is odd.} \end{cases}$$

Lemma 7.10. *For each $x \in M$, $K(x) \in M$.*

Now we show

Lemma 7.11. *$H : M \to M$ is bijective and $H^{-1} = K$.*

Proof. We show $K \circ H = \mathrm{id}_M$. By definition, $K \circ H(c) = c$.
Let $x \in M$ with $x \neq c$ and $L = L(x)$. Write

$$j = J(H(x)).$$

Case (1): $L = 0$, or $L > 0$ is even and $x_L \geq b_L$.
Subcase (1)-1: $x_L < a_L - 1$, or $x_L = a_L - 1$ with $x_{L+1} \leq b_{L+1}$.
In this subcase,

$$H(x) = (a - c)[0, L-2](a_{L-1} - \overline{\iota_{L-1}})(x_L + 1)x[L+1, \infty).$$

Suppose $L > 0$. Then $j = L - 1$ because $(a-c)_{L-1} = a_{L-1} - \iota_{L-1}$. So since j is odd with $H(x)_j \geq b_j$ by Claim 7.4 (ii), we apply the case (III) in Definition 7.8 to $H(x)$. Now since $H(x)_j = a_j - \overline{\iota_j}$ and $H(x)_{j+1} = x_L + 1 > b_L = b_{j+1}$, we have $K(H(x)) = c[0, j](H(x)_{j+1} - 1)H(x)[j+2, \infty) = c[0, L-1]x_L x[L+1, \infty) = x$.

Suppose $L = 0$. Then $H(x)_0 = x_0 + 1,\ H(x)_1 = x_1$. Moreover we have

$$j = \begin{cases} 1 \text{ if } x_0 = 0 \text{ and } \iota_0 = 1 \\ 0 \text{ otherwise.} \end{cases}$$

(Indeed, notice that $H(x)_0 = (a-c)_0$ (i.e. $x_0 + 1 = \iota_0$) $\Longleftrightarrow x_0 = 0$ and $\iota_0 = 1$. If $x_0 = 0$ and $\iota_0 = 1$, then $b_1 < a_1$ by Proposition 2.7 and so since $x \in M,\ H(x)_1 < b_1 \leq a_1 - 1 \leq (a-c)_1$, hence $j = 1$.) In case $j = 1$, we apply the case (IV) to $H(x)$ (since $H(x)_1 < b_1$) and so

$K(H(x)) = 0x[j,\infty) = x$ (since $x_0 = 0$). Next consider the case $j = 0$. Then we apply the case (I) to $H(x)$. Note that if $H(x)_0 = 1$, then $x_0 = 0$ and $\iota_0 = 0$ (because $j = 0$) and hence $H(x)_1 \geq b_1$ (since $x \in M$). Now we have $K(H(x)) = (H(x)_j - 1)H(x)[j+1,\infty) = x$.
Subcase (1)-2: $x_L = a_L - 1$ with $x_{L+1} > b_{L+1}$, or $x_L = a_L$.
In this subcase,

$$H(x) = (a-c)[0,L](x_{L+1}-1)x[L+2,\infty).$$

(Note that $x_L = a_L - \overline{\iota_L}$ because $x_L \neq c_L$.) Then

$$j = \begin{cases} L+2 \text{ if } x_{L+1} = a_{L+1} \text{ and } \iota_{L+1} = 1 \\ L+1 \text{ otherwise.} \end{cases}$$

(Indeed, $H(x)_{L+1} = (a-c)_{L+1}$ (i.e. $x_{L+1} - 1 = a_{L+1} - \iota_{L+1}$) $\Longleftrightarrow x_{L+1} = a_{L+1}$ and $\iota_{L+1} = 1$. If $x_{L+1} = a_{L+1}$ and $\iota_{L+1} = 1$, then $b_{L+2} > 0$ by Proposition 2.7 and $H(x)_{L+2} > b_{L+2} \geq 1 \geq (a-c)_{L+2}$, hence $j = L+2$.) In case $j = L+2$, we apply the case (II) to $H(x)$ (since $H(x)_{L+2} > b_{L+2}$) and so $K(H(x)) = c[0,j-3](a_{j-2} - \overline{\iota_{j-2}})a_{j-1}H(x)[j,\infty) = x$ (because $x_L = a_L - \overline{\iota_L}$ and $x_{L+1} = a_{L+1}$). Consider the case $j = L+1$. By Claim 7.4 (iii), we apply the case (III) to $H(x)$. Note that if $H(x)_{L+1} = a_{L+1} - 1$, then $x_{L+1} = a_{L+1}$ and $\iota_{L+1} = 0$ (since $j = L+1$) and so $H(x)_{L+2} = x_{L+2} \leq b_{L+2}$ (since $x \in M$). Now we have $K(H(x)) = c[0,j-2](a_{j-1} - \overline{\iota_{j-1}})(H(x)_j + 1)H(x)[j+1,\infty) = x$ (because $x_L = a_L - \overline{\iota_L}$).
Case (2): $L > 0$ is even and $x_L < b_L$.
In this case,

$$H(x) = (a-c)[0,L-3]\ \overline{\iota_{L-2}}\ 0\ x[L,\infty).$$

Then $j = L-2$ because $(a-c)_{L-2} = \iota_{L-2}$. Since $H(x)_j \leq b_j$ by Claim 7.4 (i), we apply the case (I) to $H(x)$. Note that if $H(x)_{L-2} = 1$ (that is, $\iota_{L-2} = 0$), then $x_{L-2} = c_{L-2} = a_{L-2}$ and so by Claim 7.4 (i), we have $H(x)_{j+1} = x_{L-1} - 1 \leq b_{L-1} - 1$ (because $x \in M$). Hence $K(H(x)) = c[0,j](H(x)_{j+1} + 1)H(x)[j+2,\infty) = x$.
Case (3): L is odd and $x_L \leq b_L$.
Subcase (3)-1: $x_L > 1$, or $x_L = 1$ with $x_{L+1} \geq b_{L+1}$.
In this subcase,

$$H(x) = (a-c)[0,L-2]\ \overline{\iota_{L-1}}\ (x_L - 1)x[L+1,\infty).$$

Then $j = L-1$ because $(a-c)_{L-1} = \iota_{L-1}$. We can see that $K(H(x)) = x$ by the similar argument to Subcase (1)-1 with $L > 0$.
Subcase (3)-2: $x_L = 1$ with $x_{L+1} < b_{L+1}$, or $x_L = 0$.
In this subcase,

$$H(x) = (a-c)[0,L](x_{L+1}+1)x[L+2,\infty).$$

(Note that $x_L = \overline{\iota_L}$ because $x_L \neq c_L$.) Then

$$j = \begin{cases} L+2 \text{ if } x_{L+1} = 0 \text{ and } \iota_{L+1} = 1 \\ L+1 \text{ otherwise.} \end{cases}$$

(Indeed, $H(x)_{L+1} = (a-c)_{L+1}$ (that is, $x_{L+1} + 1 = \iota_{L+1}$) $\Longleftrightarrow x_{L+1} = 0$ and $\iota_{L+1} = 1$. If $x_{L+1} = 0$ and $\iota_{L+1} = 1$, then $b_{L+2} < a_{L+2}$ by Proposition 2.7 and so since $x \in M$, $H(x)_{L+2} <$

$b_{L+2} \le a_{L+2} - 1 \le (a-c)_{L+2}$, hence $j = L+2$.) We can see that $K(H(x)) = x$ by the similar argument to Subcase (1)-2.
Case (4): L is odd and $x_L > b_L$.
In this case,

$$H(x) = (a-c)[0, L-3](a_{L-2} - \overline{\iota_{L-2}})a_{L-1}x[L,\infty).$$

In case $L > 1$, we have $j = L-2$ (because $(a-c)_{L-2} = a_{L-2} - \iota_{L-2}$) and so $K(H(x)) = x$ by the similar argument to Case (2). Consider the case $L = 1$. Then $j = 0$. So we apply the case (I) to $H(x)$. Note $H(x)_0 = a_0 = \lfloor \frac{1}{\alpha} \rfloor + \iota_0 > 1$ because $\iota_0 = 1$ by Claim 7.4 (i). Hence $K(H(x)) = (H(x)_0 - 1)H(x)[1,\infty) = x$ (since $H(x)_0 = x_0 + 1$ by Claim 7.4 (i)).
We complete the proof of $K \circ H = \mathrm{id}_M$. Similarly we can show $H \circ K = \mathrm{id}_M$. □

Proof of Theorem 1.1 (1) and (2).
Recall Remarks 6.5 and 7.6, Proposition 4.1, Lemmas 7.7 and 7.11. It suffices to show

$$\mathcal{O}_\alpha \cup \mathcal{O}_\beta \subset D := \{\xi \in [0,1) \mid \sharp\{\nu\}^{-1}(\xi) \ge 2\}.$$

Recall examples in the end of Section 3: if x is 0-left extremal or 0-right extremal, then $x \in M$ and $\{\nu\}(x) = 0$; when $\beta > 0$, if x is 1-left extremal with $x_0 = b_0$ or 1-right extremal with $x_0 = b_0 - 1$, then $x \in M$ and $\{\nu\}(x) = \beta$. Hence $\{0, \beta\} \subset D$. Since H is bijective and $\{\nu\} \circ H = R_\alpha \circ \{\nu\}$, we have $\mathcal{O}_\alpha \cup \mathcal{O}_\beta \subset D$. □

Lemma 7.12. *We have the following:*
(1) *c is left extremal $\iff$ $a-c$ is left extremal.*
(2) *c is right extremal $\iff$ $a-c$ is right extremal.*
(*Hence, c is not extremal $\iff$ $a-c$ is not extremal.*)
Moreover when c is k-left or k-right extremal,

$$b_{k+1} = \begin{cases} 0 & \text{if } k \text{ is even} \\ a_{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. (1) Let $k \in \mathbb{N}_0$. First we show that if c is k-left extremal, then for any $n \ge k$

$$\iota_n = 0,\ e_n = \begin{cases} 1 \text{ if } k \text{ is even} \\ 0 \text{ if } k \text{ is odd} \end{cases} \text{ and } b_{n+1} = \begin{cases} 0 & \text{if } k \text{ is even} \\ a_{n+1} & \text{if } k \text{ is odd.} \end{cases}$$

Since $c_k = e_k a_k$ (and $a_k \ne \iota_k$), we see that

$$\iota_k = 0 \text{ and } e_k = \begin{cases} 1 \text{ if } k \text{ is even} \\ 0 \text{ if } k \text{ is odd.} \end{cases}$$

Moreover $e_k = e_{k+2}$ because $c_{k+2} = e_{k+2}a_{k+2}$. So $e_{k+1} = |e_k - \iota_k| = e_k = e_{k+2}$. Since $b_{k+1} = b_{k+1} - (-1)^{e_k}\iota_k = c_{k+1}$ and $\iota_{k+1} = |e_{k+1} - e_{k+2}| = 0$, we have $b_{k+1} = 0$ if k is even; $b_{k+1} = a_{k+1}$ if k is odd. Now we have the desired result by Remark 2.8.

Next we show that c is k-left extremal $\implies$ $a-c$ is $(k+1)$-left extremal. By the above, we have that for each $n \geq k+1$ with $n \equiv k+1 \pmod 2$

$$(a-c)_n = \left\{\begin{matrix} a_n \text{ if } k \text{ is even} \\ 0 \ \text{ if } k \text{ is odd} \end{matrix}\right\} = e_n a_n$$

and

$$(a-c)_{n+1} = \left\{\begin{matrix} 0 \quad\ \text{ if } k \text{ is even} \\ a_{n+1} \text{ if } k \text{ is odd} \end{matrix}\right\} = b_{n+1} = b_{n+1} - (-1)^{e_n}\iota_n,$$

that is, $a-c$ is $(k+1)$-left extremal.
Similarly, we can show that $a-c$ is k-left extremal $\implies$ c is $(k+1)$-left extremal. The proof of (2) is also similar. □

Proposition 7.13. *The following conditions are equivalent*:
(1) $b_k \in \{0, a_k\}$ *for some* $k \geq 1$.
(2) $\beta \in \mathcal{O}_\alpha$.
(3) c *is extremal.*

Proof. (1) $\implies$ (2): Recall the equations in the proof of Proposition 2.7 (4), that is,

$$\beta_n = (b_n - \iota_n)\alpha_n - (-1)^{\iota_n}\beta_{n+1}\alpha_n.$$

and

$$1-\alpha_n-\beta_n = (a_n - b_n - \iota_n)\alpha_n - (-1)^{\iota_n}(1-\alpha_{n+1}-\beta_{n+1})\alpha_n.$$

(Recall $\beta_0 = \beta,\ \alpha_0 = \alpha$.) By induction on N, we can show that

$$\beta = \sum_{n=0}^{N}(-1)^n(-1)^{e_n}(b_n - \iota_n)\prod_{j=0}^{n}\alpha_j + (-1)^{N+1}(-1)^{e_{N+1}}\beta_{N+1}\prod_{j=0}^{N}\alpha_j$$
$$1-\alpha-\beta = \sum_{n=0}^{N}(-1)^n(-1)^{e_n}(a_n - b_n - \iota_n)\prod_{j=0}^{n}\alpha_j + (-1)^{N+1}(-1)^{e_{N+1}}(1-\alpha_{N+1}-\beta_{N+1})\prod_{j=0}^{N}\alpha_j$$

(recall $e_0 = 0,\ e_1 = \iota_0$ and $(-1)^{e_{n+1}} = (-1)^{e_n}(-1)^{\iota_n}$). Taking $N \to \infty$, we have

$$(i)\ \beta = \sum_{n=0}^{\infty}(-1)^n(-1)^{e_n}(b_n - \iota_n)\prod_{j=0}^{n}\alpha_j$$
$$(ii)\ 1-\alpha-\beta = \sum_{n=0}^{\infty}(-1)^n(-1)^{e_n}(a_n - b_n - \iota_n)\prod_{j=0}^{n}\alpha_j$$

by Lemma 2.6.

Suppose $b_k = 0$ for some k. Then by Remark 2.8, we have $b_n - \iota_n = 0$ ($\forall n \geq k$) and so by Lemma 2.5 and (i), $\beta \in \mathcal{O}_\alpha$.

Suppose $b_k = a_k$ for some $k \geq 1$. Then by Remark 2.8, we have $a_n - b_n - \iota_n = 0$ ($\forall n \geq k$) and so by Lemma 2.5 and (ii), $1-\alpha-\beta = q\alpha + p$ ($\exists q, p \in \mathbb{Z}$), and hence $\beta \in \mathcal{O}_\alpha$.
(2) $\implies$ (3): Firstly, consider the case $\beta = 0$ (i.e. dual Ostrowski case). In this case, $\iota_n = b_n =$

$e_n = 0$ $(\forall n)$ and so $c = a_0 0 a_2 0 \cdots$ is 0-right extremal.
Let $0 < \beta \in \mathcal{O}_\alpha$. It suffices to show if c is not right extremal, then c is left extremal. Suppose c is not right extremal. Here we use the following notations:

$$\mathbf{O}_x = \{H^n(x) \mid n \in \mathbb{Z}\} \quad \text{for } x \in M$$

and let $\mathbf{1}$ be 0-right extremal and $\mathbf{b}$ be 1-right extremal with $\mathbf{b}_0 = b_0 - 1$. So $\mathbf{1}, \mathbf{b} \in M$ and $\{\nu\}(\mathbf{1}) = 0, \{\nu\}(\mathbf{b}) = \beta$.
Since c is not right extremal, $a - c$ is also not right extremal by Lemma 7.12. Therefore since $\mathbf{1}$ is 0-right extremal, we have by the definition of H (and H^{-1})

$$\forall x \in \mathbf{O_1},\ \exists k \in \mathbb{N} : \text{even such that } x \text{ is } k\text{-right extremal.}$$

On the other hand $\{\nu\}(\mathbf{b}) = \{\nu\}(x^*)$ for some $x^* \in \mathbf{O_1}$ because $\{\nu\}(\mathbf{b}) = \beta \in \mathcal{O}_\alpha = \mathcal{O}_{\{\nu\}(\mathbf{1})} = \{\nu\}(\mathbf{O_1})$. Since $\mathbf{b}$ is right extremal, we have $\mathbf{b} = x^*$ by Lemma 6.3. Thus $\mathbf{b}$ is 1-right and k-right extremal for some even $k \in \mathbb{N}$. Then by (ii) in the proof of Lemma 6.1, we have $\iota_n = 0,\ e_n = e_k,\ b_n = \overline{e_k} a_n$ $(\forall n \geq k)$. So $c[k, \infty) = a_k 0 a_{k+2} 0 \cdots$ and we can see that if $e_k = 0$ then c is $(k+1)$-left extremal; if $e_k = 1$ then c is k-left extremal.
(3) $\Longrightarrow$ (1): by Lemma 7.12. $\square$

In particular, we have that $\beta \notin \mathcal{O}_\alpha$ if and only if $0 < b_n < a_n$ for each $n \geq 1$. In next section we use the following:

Lemma 7.14. *Let $x \in M$. Then we have*
(1) *x is left extremal $\iff$ $H(x)$ is left extremal.*
(2) *x is right extremal $\iff$ $H(x)$ is right extremal.*
(*Hence, x is not extremal $\iff$ $H(x)$ is not extremal.*)

Proof. We also use the notation $\mathbf{O}_x$ (the orbit of x under H) as above.
(1) It is sufficient to show if $x \in M$ is left extremal, then y is left extremal for each $y \in \mathbf{O}_x$. Suppose $x \in M$ is left extremal.
Case 1: c is not left extremal.
Then $a - c$ is also not left extremal by Lemma 7.12, and so we have, by the definition of H (and H^{-1}), y is left extremal for each $y \in \mathbf{O}_x$.
Case 2: c is left extremal.
Then $a - c$ is also left extremal by Lemma 7.12, and hence z is left extremal for each $z \in \mathbf{O}_c$ (by the definition of H and H^{-1}). Moreover $\beta \in \mathcal{O}_\alpha$ by Proposition 7.13. Since x is left extremal and so $\{\nu\}(x) \in \mathcal{O}_\alpha = \mathcal{O}_{\{\nu\}(c)} = \{\nu\}(\mathbf{O}_c)$, we have $x \in \mathbf{O}_c$ by Lemma 6.3, that is, $\mathbf{O}_x = \mathbf{O}_c$.
Similarly we can show (2). $\square$

§ 8. Odometer model theorem

In this section, we introduce the notion of *Denjoy systems* (cf. [4], [5]) and show the (α, β)-odometer $H : M \to M$ is topologically conjugate to a Denjoy system with cut number 1 or 2.

Let $l \in \mathbb{N}_0$ and $w = w_0 w_1 \cdots w_l \in \prod_{n=0}^{l} \{0, 1, \cdots, a_n\}$. We say w is (α, β)-**admissible** if w satisfies conditions $(1)_n, (2)_n$ in Definition 3.1 for each $0 \le n \le l-1$. For convenience' sake, we regard the empty word ϕ as an (α, β)-admissible word. When $w = w_0 w_1 \cdots w_l$ is (α, β)-admissible, define

$$[w] = \{x \in M \mid x[0, l] = w\}.$$

For each (α, β)-admissible word w, we define associated extremal sequences, l^w and r^w, as follows:

Definition 8.1 (l^w and r^w)**.** For each $k \ge 1$ and each (α, β)-admissible word $w = w_0 w_1 \cdots w_{k-1}$ of length k, define

$$L^w = \begin{cases} k & \text{if } w_{k-1} \ne e_{k-1} a_{k-1} \\ k+1 & \text{if } w_{k-1} = e_{k-1} a_{k-1}, \end{cases} \quad R^w = \begin{cases} k & \text{if } w_{k-1} \ne \overline{e_{k-1}} a_{k-1} \\ k+1 & \text{if } w_{k-1} = \overline{e_{k-1}} a_{k-1} \end{cases}$$

and let $l^w = l_0^w l_1^w \cdots$ be the L^w-left extremal sequence with

$$l^w[0, L^w - 1] = \begin{cases} w & \text{if } w_{k-1} \ne e_{k-1} a_{k-1} \\ w(b_k - (-1)^{e_{k-1}} \iota_{k-1}) & \text{if } w_{k-1} = e_{k-1} a_{k-1} \end{cases}$$

and $r^w = r_0^w r_1^w \cdots$ be the R^w-right extremal sequence with

$$r^w[0, R^w - 1] = \begin{cases} w & \text{if } w_{k-1} \ne \overline{e_{k-1}} a_{k-1} \\ w(b_k - (-1)^{\overline{e_{k-1}}} \iota_{k-1}) & \text{if } w_{k-1} = \overline{e_{k-1}} a_{k-1}. \end{cases}$$

Denote the empty word by ϕ and let l^ϕ be 0-left extremal and r^ϕ be 0-right extremal.

Lemma 8.2. *Let $k \in \mathbb{N}_0$ and $w = w_0 w_1 \cdots w_{k-1}$ be (α, β)-admissible. Then we have the following*:
(1) $\{l^w, r^w\} \subset [w]$.
(2) $\nu(l^w) < \nu(r^w)$. *Write*

$$I_w = [\nu(l^w), \nu(r^w)] \ (\subset \mathbb{R})$$

and denote its length by $|I_w|$. For any $x \in M$, we have $\lim_{l \to \infty} |I_{x[0,l]}| = 0$.
(3) *If $w w_k$ is (α, β)-admissible, then $I_{w w_k} \subset I_w$.*
(4) *If $w v_k$ and $w w_k$ are (α, β)-admissible and $v_k \ne w_k$, then $I_{w v_k} \cap \mathrm{int}\, I_{w w_k} = \emptyset$ where int I is the interior of I.*

Proof. (1) It suffices to show $\{l^w, r^w\} \subset M$. The case $w = \phi$ or $w_{k-1} \notin \{0, a_{k-1}\}$ is clear by Lemma 3.6. Consider the case $w_{k-1} = e_{k-1} a_{k-1}$. We have, by Lemma 3.6, that $l^w = w(b_k - (-1)^{e_{k-1}} \iota_{k-1}) l^w[k+1, \infty) \in M$. In order to prove $r^w \in M$, it suffices to show $r^w = w r^w[k, \infty)$ satisfies the condition $(1')_{k-1}$ in Remark 3.2. Since $\iota_{k-1} \le b_k \le a_k - \iota_{k-1}$ (by Proposition 2.7), we have $r_k^w = \overline{e_k} a_k \ge_{e_k} b_k - (-1)^{e_{k-1}} \iota_{k-1}$, that is, r^w satisfies $(1')_{k-1}$. The proof in case $w_{k-1} = \overline{e_{k-1}} a_{k-1}$ is similar.
(2) First we show $\nu(l^w) < \nu(r^w)$. By Lemma 3.5, $\nu(l^\phi) = 0 < 1 = \nu(r^\phi)$. So suppose $k \in \mathbb{N}$ and write $\nu_w = \sum_{n=0}^{k-1} \nu_n(w_n)$. When $w_{k-1} \notin \{0, a_{k-1}\}$, we have (by Lemma 3.5)

$$\nu(r^w) - \nu(l^w) = \nu_w + \overline{e_k} \prod_{j=0}^{k-1} \alpha_j - (\nu_w - e_k \prod_{j=0}^{k-1} \alpha_j) = \prod_{j=0}^{k-1} \alpha_j > 0.$$

Consider the case $w_{k-1} = e_{k-1}a_{k-1}$. Then

$$\nu(l^w) = \nu_w + \nu_k(b_k - (-1)^{e_{k-1}}\iota_{k-1}) - e_{k+1}\prod_{j=0}^{k}\alpha_j$$

$$\nu(r^w) = \nu_w + \overline{e_k}\prod_{j=0}^{k-1}\alpha_j = \nu_w + \overline{e_k}(a_k + (-1)^{\iota_k}\alpha_{k+1})\prod_{j=0}^{k}\alpha_j \quad \text{(by recursive equation (1))}.$$

So

$$\begin{aligned}&\frac{\nu(r^w) - \nu(l^w)}{\prod_{j=0}^{k}\alpha_j}\\ &= \overline{e_k}(a_k + (-1)^{\iota_k}\alpha_{k+1}) - (-1)^{e_k}(b_k - (-1)^{e_{k-1}}\iota_{k-1} - (-1)^{\iota_k}\beta_{k+1}) + e_{k+1}\\ &= \overline{e_k}a_k - (-1)^{e_k}(b_k - (-1)^{e_{k-1}}\iota_{k-1}) + \overline{e_k}(-1)^{\iota_k}\alpha_{k+1} + (-1)^{e_k}(-1)^{\iota_k}\beta_{k+1} + e_{k+1}.\end{aligned}$$

Here recall that

$$e_{k-1} = \iota_{k-1} \iff e_k = 0 \iff e_{k+1} = \iota_k$$

(by the definition: $e_n = |e_{n-1} - \iota_{n-1}|$) and that

$$-(-1)^{e_{k-1}}\iota_{k-1} = (-1)^{e_k}\iota_{k-1}$$

(because $(-1)^{e_k} = (-1)^{e_{k-1}}(-1)^{\iota_{k-1}}$ and $(-1)^s s = -s$ for each $s \in \{0,1\}$).
Therefore

$$\begin{aligned}\frac{\nu(r^w) - \nu(l^w)}{\prod_{j=0}^{k}\alpha_j} &\underset{(\flat)}{=} \begin{cases} a_k - b_k - \iota_{k-1} + (-1)^{\iota_k}\alpha_{k+1} + (-1)^{\iota_k}\beta_{k+1} + \iota_k & \text{if } e_k = 0\\ b_k - \iota_{k-1} - (-1)^{\iota_k}\beta_{k+1} + \overline{\iota_k} & \text{if } e_k = 1\end{cases}\\ &= \begin{cases} a_k - b_k - \iota_{k-1} + 1 - \left\{\dfrac{\beta_k - 1}{\alpha_k}\right\} & \text{if } e_k = 0\\ b_k - \iota_{k-1} + 1 - \left\{\dfrac{-\beta_k}{\alpha_k}\right\} & \text{if } e_k = 1\end{cases}\\ &\qquad\qquad \text{(by Remark 2.2 and Lemma 2.4)}\\ &> 0 \quad \text{(because } \iota_{k-1} \le b_k \le a_k - \iota_{k-1}\text{)},\end{aligned}$$

that is, $\nu(l^w) < \nu(r^w)$. Notice that

$$\frac{\nu(r^w) - \nu(l^w)}{\prod_{j=0}^{k}\alpha_j} = \begin{cases} \dfrac{1-\beta_k}{\alpha_k} - \iota_{k-1} & \text{if } e_k = 0\\ \dfrac{\beta_k}{\alpha_k} + \overline{\iota_{k-1}} & \text{if } e_k = 1\end{cases}$$

by the above equality $\underset{(\flat)}{=}$ and recursive equations (1) and (2). Thus

$$\nu(r^w)-\nu(l^w)=\begin{cases}(1-\beta_k-\iota_{k-1}\alpha_k)\prod_{j=0}^{k-1}\alpha_j & \text{if } e_k=0\\ (\beta_k+\overline{\iota_{k-1}}\alpha_k)\prod_{j=0}^{k-1}\alpha_j & \text{if } e_k=1\end{cases}$$

The proof in case $w_{k-1}=\overline{e_{k-1}}a_{k-1}$ is similar. Now, by Lemma 2.6, we have that $\lim_{l\to\infty}\left|I_{x[0,l]}\right|=0$ for any $x\in M$.

(3) We show $\nu(l^w)\le\nu(l^{ww_k})$. The case $w=\phi$ is clear. Suppose $k\in\mathbb{N}$.

Case 1: $w_{k-1}\neq e_{k-1}a_{k-1}$.

In this case

$$\nu(l^w)=\nu_w-e_k\prod_{j=0}^{k-1}\alpha_j.$$

If $w_k=e_ka_k$, then $l^{ww_k}=w(e_ka_k)(b_{k+1}-(-1)^{e_k}\iota_k)l^w[k+2,\infty)=l^w$ by definitions of l^{ww_k} and l^w, and so $\nu(l^{ww_k})=\nu(l^w)$.

Suppose $w_k\neq e_ka_k$. Then by Lemma 3.5 and recursive equation (1)

$$\begin{aligned}\nu(l^{ww_k})-\nu(l^w)&=\nu_k(w_k)-e_{k+1}\prod_{j=0}^{k}\alpha_j+e_k\prod_{j=0}^{k-1}\alpha_j\\&=\nu_k(w_k)-e_{k+1}\prod_{j=0}^{k}\alpha_j+e_k(a_k+(-1)^{\iota_k}\alpha_{k+1})\prod_{j=0}^{k}\alpha_j.\end{aligned}$$

Note that if $e_k=0$ then $w_k\ge 1$; if $e_k=1$ then $a_k-w_k\ge 1$. Hence

$$\begin{aligned}\frac{\nu(l^{ww_k})-\nu(l^w)}{\prod_{j=0}^{k}\alpha_j}&=\begin{cases}w_k-(-1)^{\iota_k}\beta_{k+1}-\iota_k & \text{if } e_k=0 \text{ (so } e_{k+1}=\iota_k)\\ -w_k+(-1)^{\iota_k}\beta_{k+1}-\overline{\iota_k}+a_k+(-1)^{\iota_k}\alpha_{k+1} & \text{if } e_k=1 \text{ (so } e_{k+1}=\overline{\iota_k})\end{cases}\\&=\begin{cases}w_k-\left\{\frac{-\beta_k}{\alpha_k}\right\} & \text{if } e_k=0\\ a_k-w_k-\left\{\frac{\beta_k-1}{\alpha_k}\right\} & \text{if } e_k=1\end{cases}\quad\text{(by Remark 2.2 and Lemma 2.4)}\\&>0.\end{aligned}$$

Case 2: $w_{k-1}=e_{k-1}a_{k-1}$.

In this case, since ww_k is (α,β)-admissible, we have by Remark 3.2

$$w_k\ge_{e_k} b_k-(-1)^{e_{k-1}}\iota_{k-1}\ (=b_k+(-1)^{e_k}\iota_{k-1}).$$

We show $l^{ww_k}[k+1,\infty)=l^w[k+1,\infty)$. It is clear if $w_k\neq e_ka_k$. Suppose $w_k=e_ka_k$. Then $e_ka_k=b_k+(-1)^{e_k}\iota_{k-1}=b_k-(-1)^{e_{k-1}}\iota_{k-1}$. By (i) in the proof of Lemma 6.1, we have

$\iota_n = 0,\ e_n = e_k,\ b_n = e_k a_n\ (\forall n \geq k)$. We can see that if $e_k = 0$ then $l^{ww_k}[k+1,\infty) = 000\cdots = l^w[k+1,\infty)$; if $e_k = 1$ then $l^{ww_k}[k+1,\infty) = a_{k+1}a_{k+2}a_{k+3}\cdots = l^w[k+1,\infty)$. Now we have

$$\nu(l^{ww_k}) - \nu(l^w) = \nu_k(w_k) - \nu_k(b_k - (-1)^{e_{k-1}}\iota_{k-1}) \geq 0.$$

Similarly we can show $\nu(r^{ww_k}) \leq \nu(r^w)$. Therefore $I_{ww_k} \subset I_w$.
(4) Consider the case $(-1)^{e_k}v_k < (-1)^{e_k}w_k$. Then $v_k \neq \overline{e_k}a_k$ and $w_k \neq e_k a_k$. So we have

$$\begin{aligned}\nu(l^{ww_k}) - \nu(r^{wv_k}) &= \nu_k(w_k) - e_{k+1}\prod_{j=0}^{k}\alpha_j - \nu_k(v_k) - \overline{e_{k+1}}\prod_{j=0}^{k}\alpha_j \\ &= \Big((-1)^{e_k}(w_k - v_k) - 1\Big)\prod_{j=0}^{k}\alpha_j \geq 0\end{aligned}$$

hence $I_{wv_k} \cap int\ I_{ww_k} = \emptyset$. The proof in case $(-1)^{e_k}v_k > (-1)^{e_k}w_k$ is similar. □

Now we have local version of tail inequality:

Proposition 8.3. *Let $k \in \mathbb{N}_0,\ w = w_0w_1\cdots w_{k-1}$ be (α,β)-admissible and $I_w = [\nu(l^w),\nu(r^w)]$. Then*

$$\nu([w]) = I_w \quad and \quad \nu^{-1}(int\ I_w) = [w] \setminus \{l^w, r^w\}.$$

Proof. First we show that $\nu([w]) \subset I_w$ and $\nu\Big([w]\setminus\{l^w,r^w\}\Big) \subset (\nu(l^w),\nu(r^w))$ (i.e. $[w]\setminus\{l^w,r^w\} \subset \nu^{-1}(int\ I_w)$). Let $x \in [w]$. We show that $\nu(x) \geq \nu(l^w)$ and that if $\nu(x) = \nu(l^w)$ then $x = l^w$. When $w = \phi$ or $w_{k-1} \neq e_{k-1}a_{k-1}$, by Proposition 5.2 and Lemma 3.5

$$\nu(x) - \nu(l^w) = \sum_{n=k}^{\infty}\nu_n(x_n) - \sum_{n=k}^{\infty}\nu_n(l_n^w) \geq 0$$

and if $\nu(x) = \nu(l^w)$ then $x[k,\infty) = l^w[k,\infty)$ and so $x = l^w$.
Consider the case $w_{k-1} = e_{k-1}a_{k-1}$. Then we have $x_k \geq_{e_k} b_k - (-1)^{e_{k-1}}\iota_{k-1} = l_k^w$ by Remark 3.2 (because $x \in M$ and $x_{k-1} = e_{k-1}a_{k-1}$) and so

$$\nu_k(x_k) \geq \nu_k(l_k^w).$$

On the other hand, by Proposition 5.2 and Lemma 3.5

$$\sum_{n=k+1}^{\infty}\nu_n(x_n) \geq \sum_{n=k+1}^{\infty}\nu_n(l_n^w).$$

Therefore since $\nu(x) - \nu(l^w) = \nu_k(x_k) - \nu_k(l_k^w) + \sum_{n=k+1}^{\infty}\nu_n(x_n) - \sum_{n=k+1}^{\infty}\nu_n(l_n^w)$, we have that $\nu(x) \geq \nu(l^w)$ and that if $\nu(x) = \nu(l^w)$ then $x_k = l_k^w$ and $x[k+1,\infty) = l^w[k+1,\infty)$ (by Proposition 5.2), thus $x = l^w$. Similarly we can show that $\nu(x) \leq \nu(r^w)$ and that if $\nu(x) = \nu(r^w)$ then $x = r^w$.
Next we show the following claim (recall $w = w_0w_1\cdots w_{k-1}$): for each (α,β)-admissible word v of length k,

$$v \neq w \Longrightarrow \nu([v]) \cap int\ I_w = \emptyset.$$

In case $k = 0$, there is nothing to prove and so suppose $k \in \mathbb{N}$. Let $l = \min\{n \mid v_n \neq w_n\}$ and $c = w_0 w_1 \cdots w_{l-1}$. By Lemma 8.2 (3), we have $\nu([v]) \subset I_v \subset I_{cv_l}$ and $int\ I_w \subset int\ I_{cw_l}$. Hence $\nu([v]) \cap int\ I_w \subset I_{cv_l} \cap int\ I_{cw_l} = \emptyset$ by Lemma 8.2 (4).
Next we show $\nu^{-1}(int\ I_w) \subset [w] \setminus \{l^w, r^w\}$. Let $x \in \nu^{-1}(int\ I_w)$. If $x[0, k-1] \neq w$, then $\nu(x) \notin int\ I_w$ by the above claim, and so it is a contradiction. Thus $x \in [w] \setminus \{l^w, r^w\}$.
Finally we have $\nu([w]) \supset I_w$ because $\nu : M \to [0, 1]$ is surjective (by Proposition 4.1), $\nu^{-1}(int\ I_w) \subset [w] \setminus \{l^w, r^w\}$ and $\nu(\{l^w, r^w\}) \subset \nu([w])$. □

Proof of Theorem 1.1 (3).

Let $x \in M$. By Proposition 8.3, $\nu(x) \in \nu\left(\Big[x[0, l]\Big]\right) = I_{x[0,l]}$ and moreover we have $\lim_{l \to \infty} \left|I_{x[0,l]}\right| = 0$ by Lemma 8.2 (2). Hence $\nu : M \to [0, 1]$ is continuous, and $\mathbf{e} \circ \nu : M \to S^1$ is also continuous where $\mathbf{e}(\eta) = \exp(2\pi i \eta)$. □

We recall the notion of Denjoy systems (cf. [4], [5]) and prove Theorem 1.2.
Suppose $\varphi : S^1 \to S^1$ is an orientation-preserving homeomorphism. (Naturally we identify S^1 with $[0, 1)$ via $\mathbf{e}|_{[0,1)}$.) Letting $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be a lift of φ and $\xi \in \mathbb{R}$, $\lim_{n\to\infty} \frac{\widetilde{\varphi}^n(\xi)}{n}$ exists where $\widetilde{\varphi}^n$ is the n-th iteration of $\widetilde{\varphi}$, and moreover its fractional part $\alpha \in [0, 1)$ is independent of the choices of $\widetilde{\varphi}$ and ξ. We say $\rho(\varphi) := \alpha$ is the **rotation number** of φ. One can show that $\rho(\varphi)$ is irrational if and only if φ has no periodic points. Now we can state the *Poincare's rotation number theorem*:

Suppose the rotation number α of φ is irrational. Then there is a degree 1 map $F : S^1 \to S^1$ such that $F \circ \varphi = R_\alpha \circ F$ (such F is called a **factor map** of the dynamical system (S^1, φ)). Furthermore we have the following three properties.
(1) F is unique up to rotation (i.e. when G is a factor map, $G = R_\theta \circ F$ for some θ).
Define $A = \{\xi \in S^1 \mid \sharp F^{-1}F(\xi) = 1\}$ (so $\varphi(A) = A$ and A is independent of the choice of factor maps), and let

$$X = cl\ A \quad \text{(the closure of } A\text{)}.$$

(2) The following dichotomy holds: $A = S^1$, otherwise X is a Cantor set.
We say that φ is a *Denjoy homeomorphism* if the second case holds (i.e. $A \neq S^1$). In this case, denote the restriction of φ to X by $\varphi_X : X \to X$. The subsystem (X, φ_X) is called a **Denjoy system**, and a connected component of $S^1 \setminus X$ is called a *cutout interval*; in particular, a cutout interval is an open arc.
(3) Suppose φ is a Denjoy homeomorphism. Then X is the unique minimal set under φ (here we say X is *minimal* if closed φ-invariant subset of X is $\emptyset$ or X; it is clear that the minimality of X is equivalent to the condition each φ-orbit of X is dense in X) and furthermore we have

$$X \setminus A = \{\xi \in S^1 \mid \xi \text{ is an endpoint of some cutout interval}\}$$

and $\sharp F(cl\ I) = 1$ for each cutout interval I. So, in particular, the restriction $F_X : X \to S^1$ of F to X is surjective and $\sharp F_X^{-1} F_X(\xi) = 2$ for each $\xi \notin A$. □

Let φ be a Denjoy homeomorphism. By the above third property (3), the following diagram commutes:

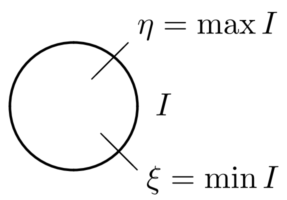
$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "F_X"'] & X \arrow[d, "F_X"] \\ S^1 \arrow[r, "R_\alpha"'] & S^1 \end{tikzcd}$$

and F_X is at most 2-to-1 surjective; precisely $\xi \in X \setminus A$ if and only if ξ is an endpoint of some cutout interval I, and in this case $F_X^{-1}F_X(\xi)$ is the set of endpoints of I.
We say $\eta \in S^1$ is a **double point** of F_X if $\eta \in F_X(X \setminus A)$. Since there is countably many cutout intervals, the set $F_X(X \setminus A)$ of double points is countable and $R_\alpha(F_X(X \setminus A)) = F_X(X \setminus A)$. Therefore there is $d \in \mathbb{N} \cup \{\infty\}$ such that

$$F_X(X \setminus A) = \bigcup_{k=1}^{d} \mathcal{O}_{\eta_k} \ (disjoint) \ \text{ for some } \{\eta_k\}_{k=1}^{d} \subset S^1$$

in other words, the set of double points is split into at most countably many R_α-orbits (by the above first property (1), we can suppose $\eta_1 = \alpha$ without the loss of generality). Moreover note that the cardinality d is independent of the choice of F_X's. We call d the **cut number** of φ (or φ_X).

For each closed arc $I \subset S^1$, we write $I = [\xi, \eta]$ where ξ (resp. η) is the minimum (resp. maximum) of I in circular order (that is, the counterclockwise orientation of S^1) and so write $\xi = \min I,\ \eta = \max I$ and $int\ I = (\xi, \eta)$ where $int\ I$ is the interior of I:



Remark 8.4. *Let $\xi, \eta \in S^1$ be distinct double points of F_X (so $F^{-1}(\xi)$ and $F^{-1}(\eta)$ are disjoint closed arcs). Define $\widetilde{\xi} = \max F^{-1}(\xi),\ \widetilde{\eta} = \min F^{-1}(\eta)$ and let $I = [\xi, \eta]$. Then $\{\widetilde{\xi}, \widetilde{\eta}\} \cup F_X^{-1}(int\ I) = [\widetilde{\xi}, \widetilde{\eta}] \cap X$. (So, in particular, $\{\widetilde{\xi}, \widetilde{\eta}\} \cup F_X^{-1}(int\ I)$ is closed.)*

Proof. Since $F : S^1 \to S^1$ is degree 1 (hence F is (continuous) monotone non-decreasing), we have $F^{-1}(int\ I) = (\widetilde{\xi}, \widetilde{\eta})$. So $F_X^{-1}(int\ I) = (\widetilde{\xi}, \widetilde{\eta}) \cap X$. □

Proof of Theorem 1.2.
Let (X, φ_X) be a Denjoy system with rotation number α and a factor map F which satisfies $F_X(X \setminus A) = \mathcal{O}_\alpha \cup \mathcal{O}_\beta$ where we identify F with $(\mathbf{e}|_{[0,1)})^{-1} \circ F : S^1 \to [0, 1)$. Define

$$E = \{x \in M \mid x : \text{extremal}\} \ \text{ and } \ N = M \setminus E.$$

Recall that the restriction, $F_A : A \to [0, 1) \setminus \mathcal{O}_\alpha \cup \mathcal{O}_\beta$, of F to A is bijective and that $\mathcal{O}_\alpha \cup \mathcal{O}_\beta = \{\nu\}(E)$ and the restriction, $\{\nu\}_N : N \to [0, 1) \setminus \mathcal{O}_\alpha \cup \mathcal{O}_\beta$, of $\{\nu\}$ to N is also bijective (by Theorem 1.1).

Define $\psi : X \to M$ in the following way. For each $\xi \in A$, define

$$\psi(\xi) = \{\nu\}_N^{-1} \circ F(\xi).$$

Let $\xi \in X \setminus A$. Then $F(\xi) \in \mathcal{O}_\alpha \cup \mathcal{O}_\beta$ and $F^{-1}F(\xi)$ is the closure of a cutout interval. So by the argument in the proof of Theorem 1.1 (1) and (2), we have $F(\xi) = \{\nu\}(x) = \{\nu\}(y)$ for some doubleton $\{x, y\}$ where x is left extremal and y is right extremal. Define

$$\psi(\xi) = \begin{cases} x \text{ if } \xi = \max F^{-1}F(\xi) \\ y \text{ if } \xi = \min F^{-1}F(\xi). \end{cases}$$

First we show $\psi : X \to M$ is bijective. Indeed, we can naturally define the inverse $\psi^{-1} : M \to X$ as follows. For each $x \in N$, define

$$\psi^{-1}(x) = F_A^{-1} \circ \{\nu\}(x)$$

and for each $x \in E$, define

$$\psi^{-1}(x) = \begin{cases} \max F^{-1}\{\nu\}(x) \text{ if } x \text{ is left extremal} \\ \min F^{-1}\{\nu\}(x) \text{ if } x \text{ is right extremal.} \end{cases}$$

(Note $\{\nu\} \circ \psi = F_X,\ \psi(A) = N$ and $\psi(X \setminus A) = E$ by definition.)

Next we show $\psi \circ \varphi = H \circ \psi$. For each $\xi \in A$,

$$\psi \circ \varphi(\xi) = \{\nu\}_N^{-1} \circ F \circ \varphi(\xi) = \{\nu\}_N^{-1} \circ R_\alpha \circ F(\xi) = H \circ \{\nu\}_N^{-1} \circ F(\xi) = H \circ \psi(\xi).$$

Since $\varphi : S^1 \to S^1$ is orientation-preserving, notice that $\xi = \max F^{-1}F(\xi)$ if and only if $\varphi(\xi) = \max F^{-1}F\varphi(\xi)$ for each $\xi \in X \setminus A$. For each $x \in E$, by Lemma 7.14, x is left extremal if and only if $H(x)$ is left extremal. Hence, for each $\xi \in X \setminus A$, we have that $\psi \circ \varphi(\xi)$ is left extremal if and only if $H \circ \psi(\xi)$ is left extremal. Since $\{\nu\} \circ \psi \circ \varphi(\xi) = F \circ \varphi(\xi) = R_\alpha \circ F(\xi) = R_\alpha \circ \{\nu\} \circ \psi(\xi) = \{\nu\} \circ H \circ \psi(\xi)$, we have (by Lemma 6.3) $\psi \circ \varphi(\xi) = H \circ \psi(\xi)$ for each $\xi \in X \setminus A$.

Finally we show $\psi : X \to M$ is continuous. It suffices to show that $\psi^{-1}([w]) \subset X$ is open for each (α, β)-admissible word w of length $k \geq 1$. At first, we show $\psi^{-1}([w])$ is closed. By Proposition 8.3, we have

$$\{\nu\}^{-1}(int\ I_w) = [w] \setminus \{l^w, r^w\}.$$

So

$$F_X^{-1}(int\ I_w) = \psi^{-1}\{\nu\}^{-1}(int\ I_w) = \psi^{-1}([w] \setminus \{l^w, r^w\}) = \psi^{-1}([w]) \setminus \{\psi^{-1}(l^w), \psi^{-1}(r^w)\}.$$

Here, regarding $I_w = [\nu(l^w), \nu(r^w)]$ as a closed arc in S^1, the closed arc I_w has $\{\nu\}(l^w)$ as its minimum and $\{\nu\}(r^w)$ as its maximum. Since $\psi^{-1}(l^w) = \max F^{-1}\{\nu\}(l^w)$ and $\psi^{-1}(r^w) = \min F^{-1}\{\nu\}(r^w)$, we have, by Remark 8.4,

$$\psi^{-1}([w]) = \{\psi^{-1}(l^w), \psi^{-1}(r^w)\} \cup F_X^{-1}(int\ I_w) \text{ is closed.}$$

Since

$$\psi^{-1}([w]) = X \setminus \bigcup \{\psi^{-1}([v]) \mid v : (\alpha, \beta)\text{-admissible of length } k \text{ with } v \neq w\},$$

$\psi^{-1}([w])$ is open in X. $\square$

§ 9. Appendix: Proof of Lemma 2.5 and Lemma 2.6

First recall basic properties of general continued fractions. We use the following notation:

$$\cfrac{B_0}{A_0 + \cfrac{B_1}{A_1 + \ddots + \cfrac{B_n}{A_n}}} = \frac{B_0|}{|A_0} + \frac{B_1|}{|A_1} + \cdots + \frac{B_n|}{|A_n}.$$

Definition 9.1. Define sequences $\{Q_n\}_{n\geq -2}$ and $\{P_n\}_{n\geq -2}$ by

$$\begin{pmatrix} P_{-2} & P_{-1} \\ Q_{-2} & Q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for $n \geq 0$,

$$P_n = A_n P_{n-1} + B_n P_{n-2}$$
$$Q_n = A_n Q_{n-1} + B_n Q_{n-2}.$$

We call $\{Q_n\}_{n\geq -2}$ and $\{P_n\}_{n\geq -2}$ the sequences associated with $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$.

Claim 9.2. For each $n \geq 0$

$$\frac{P_n}{Q_n} = \frac{B_0|}{|A_0} + \frac{B_1|}{|A_1} + \cdots + \frac{B_n|}{|A_n}.$$

Proof. It suffices to show that for each $n \geq 0$

$$\frac{A_n P_{n-1} + B_n P_{n-2}}{A_n Q_{n-1} + B_n Q_{n-2}} = \frac{B_0|}{|A_0} + \frac{B_1|}{|A_1} + \cdots + \frac{B_n|}{|A_n}.$$

Indeed, it is clear when $n = 0$. Now suppose the above statement holds for n. Then

$$\begin{aligned} \frac{B_0|}{|A_0} + \frac{B_1|}{|A_1} + \cdots + \frac{B_{n+1}|}{|A_{n+1}} &= \frac{B_0|}{|A_0} + \frac{B_1|}{|A_1} + \cdots + \frac{B_{n-1}|}{|A_{n-1}} + \frac{B_n \quad |}{\left|A_n + \dfrac{B_{n+1}}{A_{n+1}}\right.} \\ &= \frac{(A_n + \frac{B_{n+1}}{A_{n+1}})P_{n-1} + B_n P_{n-2}}{(A_n + \frac{B_{n+1}}{A_{n+1}})Q_{n-1} + B_n Q_{n-2}} \\ &= \frac{P_n + \frac{B_{n+1}}{A_{n+1}} P_{n-1}}{Q_n + \frac{B_{n+1}}{A_{n+1}} Q_{n-1}} \\ &= \frac{A_{n+1} P_n + B_{n+1} P_{n-1}}{A_{n+1} Q_n + B_{n+1} Q_{n-1}}. \end{aligned}$$

So by induction on n, we have the desired result. $\square$

Claim 9.3. For each $n \geq 0$

$$Q_n P_{n-1} - Q_{n-1} P_n = (-1)^{n+1} B_0 B_1 \cdots B_n.$$

Proof. First note that for each $n \geq 0$

$$\begin{pmatrix} Q_n & Q_{n-1} \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} Q_{n-1} & Q_{n-2} \\ P_{n-1} & P_{n-2} \end{pmatrix} \begin{pmatrix} A_n & 1 \\ B_n & 0 \end{pmatrix}.$$

So we have for each $n \geq 0$,

$$\begin{pmatrix} Q_n & Q_{n-1} \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} A_0 & 1 \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & 1 \\ B_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} A_n & 1 \\ B_n & 0 \end{pmatrix}.$$

By taking determinants, we obtain the claim. $\square$

Claim 9.4. Let $B_0 = 1$, and suppose that $\{\gamma_n\}_{n\geq 0} \subset \mathbb{R}$ satisfies the following conditions:

$$A_n\gamma_n + B_{n+1}\gamma_{n+1}\gamma_n = 1 \quad (n = 0, 1, \dots).$$

Then for each $n \geq 0$, we have

$$(1)\ \gamma_0(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1}) = P_n + P_{n-1}B_{n+1}\gamma_{n+1}$$

$$(2)\ Q_n\gamma_0 - P_n = (-1)^{n+1}B_1 \cdots B_{n+1}\gamma_{n+1}\gamma_n \cdots \gamma_0$$

$$(3)\ \gamma_0 - \frac{P_n}{Q_n} = \frac{(-1)^{n+1}B_1 \cdots B_{n+1}\gamma_{n+1}}{Q_n(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1})}$$

$$(4)\ \gamma_n \cdots \gamma_0 = \frac{1}{Q_n + Q_{n-1}B_{n+1}\gamma_{n+1}}.$$

Proof. We show the following statement: for each $n \geq 0$

$$Q_n\gamma_0 - P_n = -B_{n+1}\gamma_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}).$$

Indeed, we show by induction on n. First

$$Q_0\gamma_0 - P_0 = A_0\gamma_0 - B_0 = 1 - B_1\gamma_1\gamma_0 - B_0 = -B_1\gamma_1\gamma_0 = -B_1\gamma_1(Q_{-1}\gamma_0 - P_{-1}).$$

Suppose the above statement holds for n. Then we have

$$\begin{aligned}
Q_{n+1}\gamma_0 - P_{n+1} &= (A_{n+1}Q_n + B_{n+1}Q_{n-1})\gamma_0 - (A_{n+1}P_n + B_{n+1}P_{n-1}) \\
&= A_{n+1}(Q_n\gamma_0 - P_n) + B_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}) \\
&= -A_{n+1}B_{n+1}\gamma_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}) + B_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}) \\
&= (-A_{n+1}\gamma_{n+1} + 1)B_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}) \\
&= B_{n+2}\gamma_{n+2}\gamma_{n+1}B_{n+1}(Q_{n-1}\gamma_0 - P_{n-1}) \\
&= -B_{n+2}\gamma_{n+2}(Q_n\gamma_0 - P_n),
\end{aligned}$$

that is, the above statement also holds for $n + 1$.

Now (1) follows the above statement. Since $Q_{-1}\gamma_0 - P_{-1} = \gamma_0$, (2) also follows the above. So by (1) and Claim 9.3, we have

$$\begin{aligned}
\gamma_0 - \frac{P_n}{Q_n} &= \frac{P_n + P_{n-1}B_{n+1}\gamma_{n+1}}{Q_n + Q_{n-1}B_{n+1}\gamma_{n+1}} - \frac{P_n}{Q_n} \\
&= \frac{Q_n(P_n + P_{n-1}B_{n+1}\gamma_{n+1}) - P_n(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1})}{Q_n(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1})} \\
&= \frac{(Q_nP_{n-1} - P_nQ_{n-1})B_{n+1}\gamma_{n+1}}{Q_n(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1})} \\
&= \frac{(-1)^{n+1}B_1 \cdots B_{n+1}\gamma_{n+1}}{Q_n(Q_n + Q_{n-1}B_{n+1}\gamma_{n+1})},
\end{aligned}$$

thus (3) holds. Finally (4) follows (2) and (3). □

Now we prove Lemmas 2.5 and 2.6.
Recall definitions of $\{a_n\}_{n\ge 0}$, $\{\iota_n\}_{n\ge -1}$ and $\{\alpha_n\}_{n\ge 0}$.

Proof of Lemma 2.5.
Let $A_n = a_n$ and $B_n = (-1)^{\iota_{n-1}}$ for each $n \ge 0$ (in particular $B_0 = 1$ since $\iota_{-1} = 0$). Then by recursive equation (1)

$$A_n\alpha_n + B_{n+1}\alpha_{n+1}\alpha_n = 1.$$

So letting $\{Q_n\}_{n\ge -2}$ and $\{P_n\}_{n\ge -2}$ be sequences associated with $\{A_n\}_{n\ge 0}$ and $\{B_n\}_{n\ge 0}$, we have by Claim 9.4 (2)

$$\prod_{j=0}^{n+1} \alpha_j = (-1)^{n+1}(-1)^{\iota_0+\iota_1+\cdots+\iota_n}(Q_n\alpha - P_n)$$

for each $n \ge 0$. □

In order to show Lemma 2.6, we need the following two propositions.

Proposition 9.5. *Let $N \in \mathbb{N}_0$. For each $n \ge 0$, define*

$$A_n = a_{N+n}$$

and

$$B_0 = 1,\ B_n = (-1)^{\iota_{N+n-1}} \quad (n \ge 1).$$

Let $\{Q_n\}_{n\ge -2}$ and $\{P_n\}_{n\ge -2}$ be the sequences associated with $\{A_n\}_{n\ge 0}$ and $\{B_n\}_{n\ge 0}$. Then

$$Q_{n-1} < Q_n \ (\forall n \ge 1), \quad \lim_{n\to\infty} Q_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{P_n}{Q_n} = \alpha_N.$$

Proof. First (recall $a_n \ge 1$ and) notice that if $\iota_{n-1} = 1$ or $\iota_n = 1$, then $a_n \ge 2$. Indeed $a_n = \left\lfloor \frac{1}{\alpha_n} \right\rfloor + \iota_n \ge 1 + \iota_n$ and moreover, by Proposition 2.7, we have $a_n \ge b_n + \iota_{n-1} \ge 2\iota_{n-1}$.

Next we show that $\{Q_n\}_{n\geq 0}$ is strictly increasing sequence in $\mathbb{N}$ (therefore $\lim_{n\to\infty} Q_n = \infty$). Indeed, by induction, we show $Q_n > Q_{n-1} \geq 1$ for each $n \geq 1$. Firstly $Q_0 = a_N \geq 1$ and so

$$Q_1 - Q_0 = (a_{N+1} - 1)Q_0 + (-1)^{\iota_N} Q_{-1} = (a_{N+1} - 1)a_N + (-1)^{\iota_N}.$$

So if $\iota_N = 0$ then $Q_1 - Q_0 \geq (-1)^{\iota_N} = 1$; if $\iota_N = 1$ then $Q_1 - Q_0 \geq a_N + (-1)^{\iota_N} \geq 1$.
Let $n \geq 2$ and suppose $Q_{n-1} > Q_{n-2} \geq 1$. Here

$$Q_n - Q_{n-1} = (a_{N+n} - 1)Q_{n-1} + (-1)^{\iota_{N+n-1}} Q_{n-2}.$$

So if $\iota_{N+n-1} = 0$ then $Q_n - Q_{n-1} \geq (-1)^{\iota_{N+n-1}} Q_{n-2} = Q_{n-2} \geq 1$; if $\iota_{N+n-1} = 1$ then $Q_n - Q_{n-1} \geq Q_{n-1} + (-1)^{\iota_{N+n-1}} Q_{n-2} = Q_{n-1} - Q_{n-2} \geq 1$.
Next, define for each $n \geq 0$

$$\gamma_n = \alpha_{N+n}.$$

Then $\{\gamma_n\}_{n\geq 0}$ satisfies the assumption in Claim 9.4 (by recursive equation (1)). Hence by Claim 9.4 (3), we have for each $n \geq 1$

$$\alpha_N - \frac{P_n}{Q_n} = \frac{(-1)^{n+1}(-1)^{\iota_N + \iota_{N+1} + \cdots + \iota_{N+n}} \alpha_{N+n+1}}{Q_n(Q_n + Q_{n-1}(-1)^{\iota_{N+n}} \alpha_{N+n+1})}$$

and so (since $Q_n - Q_{n-1} \geq 1$ and $0 < \alpha_{N+n+1} < 1$)

$$\left| \alpha_N - \frac{P_n}{Q_n} \right| = \frac{\alpha_{N+n+1}}{Q_n(Q_n + Q_{n-1}(-1)^{\iota_{N+n}} \alpha_{N+n+1})} < \frac{1}{Q_n}.$$

Hence $\lim_{n\to\infty} \dfrac{P_n}{Q_n} = \alpha_N$. □

Note. In particular, by Proposition 9.5 and Claim 9.2, we have the semi-regular continued fraction expansion of α:

$$\alpha = \cfrac{1}{a_0 + \cfrac{(-1)^{\iota_0}}{a_1 + \cfrac{(-1)^{\iota_1}}{a_2 + \ddots}}}.$$

Proposition 9.6. *Let $N \in \mathbb{N}_0$. If $\iota_n = 1$ for each $n \geq N$, then $a_{n_0} \geq 3$ for some $n_0 \geq N$.*

Proof. Note $a_n \geq 2$ for each $n \geq N$ (because $a_n = \left\lfloor \frac{1}{\alpha_n} \right\rfloor + \iota_n$). We show by contradiction. Assume $a_n = 2$ for each $n \geq N$. Following the setup in Proposition 9.5, define

$$A_n = a_{N+n} = 2 \quad (n \geq 0)$$

and

$$B_0 = 1,\ B_n = (-1)^{\iota_{N+n-1}} = -1 \ \ (n \geq 1)$$

and let $\{Q_n\}_{n\geq -2}$ and $\{P_n\}_{n\geq -2}$ be the sequences associated with $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$. So by Proposition 9.5

$$\alpha_N = \lim_{n\to\infty} \frac{P_n}{Q_n}.$$

On the other hand, by Claim 9.2, for each $n \geq 0$

$$\frac{P_n}{Q_n} = \frac{1|}{|2} + \underbrace{\frac{-1|}{|\ 2} + \cdots + \frac{-1|}{|\ 2}}_{n \text{ times}}.$$

Moreover we show for each $n \geq 0$

$$\frac{P_n}{Q_n} = \frac{n+1}{n+2}.$$

Indeed, it is clear for $n = 0$. Let $n \geq 0$, and suppose that $\frac{P_n}{Q_n} = \frac{n+1}{n+2}$. Then by the above representation of $\frac{P_{n+1}}{Q_{n+1}}$ in finite continued fraction form, we have

$$2 - \left(\frac{P_{n+1}}{Q_{n+1}}\right)^{-1} = \frac{1|}{|2} + \underbrace{\frac{-1|}{|\ 2} + \cdots + \frac{-1|}{|\ 2}}_{n \text{ times}} = \frac{P_n}{Q_n} = \frac{n+1}{n+2}$$

and so $\frac{P_{n+1}}{Q_{n+1}} = \frac{n+2}{n+3}$.
Therefore

$$\alpha_N = \lim_{n\to\infty} \frac{P_n}{Q_n} = 1$$

contradicting $\alpha_N < 1$. □

Proof of Lemma 2.6.

First note that $\{\alpha_n \cdots \alpha_0\}_{n\geq 0}$ is a strictly decreasing sequence in $(0, 1)$. So, in order to prove $\lim_{n\to\infty} \alpha_n \cdots \alpha_0 = 0$, it suffices to show there is a subsequence converging to zero. Following the setup in Proposition 9.5, define

$$A_n = a_n \quad (n \geq 0)$$

and

$$B_n = (-1)^{\iota_{n-1}} \quad (n \geq 0)$$

(in particular $B_0 = 1$ since $\iota_{-1} = 0$) and let $\{Q_n\}_{n\geq -2}$ and $\{P_n\}_{n\geq -2}$ be the sequences associated with $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$. Then we have by Claim 9.4 (4), for each $n \geq 0$

$$\alpha_n \cdots \alpha_0 = \frac{1}{Q_n + Q_{n-1}(-1)^{\iota_n}\alpha_{n+1}}.$$

Here let $N = \{n \in \mathbb{N}_0 \mid \iota_n = 0\}$.
Case 1: $\sharp N = \infty$.
For each $n \in N$

$$\alpha_n \cdots \alpha_0 = \frac{1}{Q_n + Q_{n-1}\alpha_{n+1}} < \frac{1}{Q_n}.$$

Hence the subsequence $\{\alpha_n \cdots \alpha_0\}_{n\in N}$ converges to zero.
Case 2: $\sharp N < \infty$.
Then, letting $L = \{n \in \mathbb{N}_0 \mid a_n \geq 3\}$, we have $\sharp L = \infty$ by Proposition 9.6. For each $n \in L$

$$Q_n + Q_{n-1}(-1)^{\iota_n}\alpha_{n+1} = a_n Q_{n-1} + (-1)^{\iota_{n-1}} Q_{n-2} + Q_{n-1}(-1)^{\iota_n}\alpha_{n+1} > (a_n - 2)Q_{n-1} \geq Q_{n-1}$$

and so

$$\alpha_n \cdots \alpha_0 = \frac{1}{Q_n + Q_{n-1}(-1)^{\iota_n}\alpha_{n+1}} < \frac{1}{Q_{n-1}}.$$

Hence the subsequence $\{\alpha_n \cdots \alpha_0\}_{n \in L}$ converges to zero. $\square$

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References

[1] M.I. Cortez and J. Rivera-Letelier: *Topological orbit equivalence classes and numeration scales of logistic maps* Erg.Th. and Dynam.Sys. **32**, pp.1501-1526 (2012).

[2] G. Didier: *Codages de rotations et fractions continues.* J. Number Theory **71**, pp.275-306 (1998).

[3] N.P. Fogg: Substitutions in Dynamics, Arithmetics and Combinatorics. Lecture Notes in Math. Vol.1794, Springer (2002).

[4] B.A. Itza-Ortiz and N.C. Phillips: *Realization of a simple higher-dimensional noncommutative torus as a transformation group C^*-algebra.* Bull. London Math. Soc. **40**, pp.217-226 (2008).

[5] K. Masui, F. Sugisaki and M. Yoshida *Denjoy systems and dimension groups.* Erg.Th. and Dynam.Sys. **28**, pp.877-913 (2008).